

# Lifting mixing properties by Rokhlin cocycles

M. Lemańczyk\*      F. Parreau

## Abstract

We study the problem of lifting various mixing properties from a base automorphism  $T \in \text{Aut}(X, \mathcal{B}, \mu)$  to skew products of the form  $T_{\varphi, \mathcal{S}}$ , where  $\varphi : X \rightarrow G$  is a cocycle with values in a locally compact Abelian group  $G$ ,  $\mathcal{S} = (S_g)_{g \in G}$  is a measurable representation of  $G$  in  $\text{Aut}(Y, \mathcal{C}, \nu)$  and  $T_{\varphi, \mathcal{S}}$  acts on the product space  $(X \times Y, \mathcal{B} \otimes \mathcal{C}, \mu \otimes \nu)$  by

$$T_{\varphi, \mathcal{S}}(x, y) = (Tx, S_{\varphi(x)}(y)).$$

It is also shown that whenever  $T$  is ergodic (mildly mixing, mixing) but  $T_{\varphi, \mathcal{S}}$  is not ergodic (is not mildly mixing, not mixing), then on a non-trivial factor  $\mathcal{A} \subset \mathcal{C}$  of  $\mathcal{S}$  the corresponding Rokhlin cocycle  $x \mapsto S_{\varphi(x)}|_{\mathcal{A}}$  is a coboundary (a quasi-coboundary).

## Introduction

Given an ergodic automorphism  $T$  of a standard Borel space  $(X, \mathcal{B}, \mu)$  we can study various extensions  $\tilde{T}$  of it. Among such extensions a special role is played by so called compact group extensions or, more generally, isometric extensions (see [8], [12] and [30]). In particular, one can ask which ergodic properties of  $T$  are lifted by isometric extensions. The two papers<sup>1</sup> by Dan Rudolph [25] and [26] are beautiful examples of the mechanism that once the extension enjoys some “minimal” ergodic property then it shares some strong ergodic properties assumed to hold for its base. By iterating the procedure of taking isometric extensions we can hence lift ergodic properties of  $T$  to weakly mixing distal extensions of it.

---

\*Research partly supported by Polish MNiSzW grant N N201 384834

<sup>1</sup>In [25] it is proved that Bernoullicity is lifted whenever the extension is weakly mixing, while in [26] it is shown that mixing (multiple mixing) lifts whenever the extension is weakly mixing.

The notion complementary to distality is relative weak mixing [8], [12], [30] and a natural question arises what happens with lifting ergodic properties from  $T$  to  $\tilde{T}$  when  $\tilde{T}$  is relatively weakly mixing over the factor  $T$ . This, by Abramov-Rokhlin's theorem [2], leads to the study of so called Rokhlin cocycle extensions which are automorphisms of the form  $\tilde{T} = T_\Theta$  acting on  $(X \times Y, \mathcal{B} \otimes \mathcal{C}, \mu \otimes \nu)$  by the formula

$$T_\Theta(x, y) = (Tx, \Theta_x(y)),$$

where  $\Theta : X \rightarrow \text{Aut}(Y, \mathcal{C}, \nu)$  is measurable<sup>2</sup>. Since the above formula describes all possible (ergodic) extensions of  $T$ , it is hard to expect interesting theorems on such a level of generality – one has to specify subclasses of Rokhlin cocycles for which one can obtain some results. We will focus on the following class.

Let  $G$  be a second countable locally compact Abelian (LCA) group. Assume that we have a measurable action  $\mathcal{S}$  of this group given by  $g \mapsto S_g \in \text{Aut}(Y, \mathcal{C}, \nu)$ . Let  $\varphi : X \rightarrow G$  be a cocycle. The automorphism  $T_{\varphi, \mathcal{S}}$  acting on  $(X \times Y, \mathcal{B} \otimes \mathcal{C}, \mu \otimes \nu)$  given by

$$T_{\varphi, \mathcal{S}}(x, y) = (Tx, S_{\varphi(x)}(y))$$

will be called the *Rokhlin  $(\varphi, \mathcal{S})$ -extension*<sup>3</sup> of  $T$ .

A systematic study of the problem of lifting ergodic properties from  $T$  to  $T_{\varphi, \mathcal{S}}$  was originated by D. Rudolph in [27]. Since then, extensions  $T_{\varphi, \mathcal{S}} \rightarrow T$  have been studied in numerous papers, see *e.g.* [5], [11], [12], [13], [21], [22], [24] and [28].

The present paper is a continuation of investigations from [21] and [22], and, due to a new approach presented here, makes them complete. This new approach is based on a harmonic analysis result from [17], and it consists in showing that given an action  $\mathcal{S} = (S_g)_{g \in G}$  of a second countable LCA group  $G$  on a probability standard Borel space  $(Y, \mathcal{C}, \nu)$  and a saturated Borel subgroup  $\Lambda \subset \widehat{G}$ , the spectral space of functions in  $L^2(Y, \mathcal{C}, \nu)$  whose spectral measures are concentrated on  $\Lambda$  is the  $L^2$ -space of an  $\mathcal{S}$ -invariant sub- $\sigma$ -algebra  $\mathcal{A} \subset \mathcal{C}$  (a measure-theoretic factor of  $\mathcal{S}$ ). This will systematically be used in our study because the group of  $L^\infty$ -eigenvalues of the Mackey  $G$ -action associated to  $T$  and  $\varphi$  is saturated and hence yields an  $\mathcal{S}$ -factor.

---

<sup>2</sup>The map  $\Theta$  is often called a Rokhlin cocycle.

<sup>3</sup>We would like to emphasize that, as noticed in [5], if we admit  $G$  to be non-Abelian locally compact, then each ergodic extension  $\tilde{T} = T_\Theta$  is of the form  $T_{\varphi, \mathcal{S}}$ ; more specifically, a general Rokhlin cocycle  $x \mapsto \Theta(x)$  is cohomologous to a cocycle  $x \mapsto S_{\varphi(x)}$  for some  $G, \varphi$  and  $\mathcal{S}$ .

Using that we will prove natural necessary and sufficient conditions for weak mixing of  $T_{\varphi, \mathcal{S}}$  and relative weak mixing of  $T_{\varphi, \mathcal{S}}$  over  $T$ . We also compute possible eigenvalues of  $T_{\varphi, \mathcal{S}}$  and determine the relative Kronecker factor whenever  $T_{\varphi, \mathcal{S}}$  is ergodic. The idea of a factor determined by a saturated group allows us to prove that if  $T$  is ergodic but  $T_{\varphi, \mathcal{S}}$  is not, then the Rokhlin cocycle  $x \mapsto S_{\varphi(x)}|_{\mathcal{A}}$  is a coboundary as a cocycle taking values in  $\text{Aut}(\mathcal{A})$ , where  $\mathcal{A}$  is the non-trivial factor of  $\mathcal{S}$  corresponding to the above-mentioned eigenvalue group. Finally, by replacing coboundary by quasi-coboundary, a similar conclusion is achieved when  $T$  is mildly mixing but  $T_{\varphi, \mathcal{S}}$  is not, and when  $T$  is mixing but  $T_{\varphi, \mathcal{S}}$  is not.

Another tool explored here is a use of mixing sequences of weighted unitary operators, that is, of operators on  $L^2(X, \mathcal{B}, \mu)$  given by the formula

$$f \mapsto \xi \cdot f \circ T \text{ for each } f \in L^2(X, \mathcal{B}, \mu)$$

determined by a measurable  $\xi : X \rightarrow \mathbb{T}$  and an automorphism  $T$ . This, in particular, will solve the problem of lifting mild mixing property, and complete the picture from [22] of lifting mixing and multiple mixing.

## 1 Preliminaries

We briefly recall basic definitions, some known results and fix notation for the rest of the paper.

### 1.1 Self-joinings of an automorphism, relative concepts

Assume that  $T$  is an automorphism of a standard probability Borel space  $(X, \mathcal{B}, \mu)$ , which we denote  $T \in \text{Aut}(X, \mathcal{B}, \mu)$ <sup>4</sup>. Denote by  $J(T)$  the set of self-joinings of  $T$ , that means the set of  $T \times T$ -invariant probability measures on  $(X \times X, \mathcal{B} \otimes \mathcal{B})$  whose both marginals are equal to  $\mu$ . To each self-joining  $\eta \in J(T)$  one associates a Markov operator<sup>5</sup>  $\Phi_\eta$  of  $L^2(X, \mathcal{B}, \mu)$  given by

$$\int_X \Phi_\eta f(y) g(y) d\mu(y) = \int_{X \times X} f(x) g(y) d\eta(x, y)$$

for each  $f, g \in L^2(X, \mathcal{B}, \mu)$ . Moreover, the  $T \times T$ -invariance of  $\eta$  means that

$$\Phi_\eta \circ T = T \circ \Phi_\eta. \tag{1}$$

---

<sup>4</sup>We shall also denote by  $T$  the unitary operator  $f \mapsto f \circ T$  on  $L^2(X, \mathcal{B}, \mu)$ .

<sup>5</sup>A linear bounded operator  $\Phi$  of  $L^2(X, \mathcal{B}, \mu)$  is called *Markov* if  $\Phi(1) = 1 = \Phi^*(1)$  and  $\Phi f \geq 0$  whenever  $f \geq 0$ . Notice also that we always have  $\|\Phi_\eta f\| \leq \|f\|$  and thus  $\|\Phi_\eta\| = 1$ .

On the other hand each Markov operator  $\Phi$  on  $L^2(X, \mathcal{B}, \mu)$  for which (1) holds determines a self-joining  $\eta_\Phi$  by the formula

$$\eta_\Phi(A \times B) = \int_B \Phi(1_A) d\mu$$

for each  $A, B \in \mathcal{B}$ . Then

$$\Phi = \Phi_{\eta_\Phi} \text{ and } \eta = \eta_{\Phi_\eta}. \quad (2)$$

Therefore the set  $J(T)$  can naturally be identified with the set  $\mathcal{J}(T)$  of Markov operators on  $L^2(X, \mathcal{B}, \mu)$  satisfying (1). The set  $\mathcal{J}(T)$  is a closed subset in the weak operator topology and hence it is compact. Thus

$$\Phi_n \rightarrow \Phi \text{ iff } \langle \Phi_n f, g \rangle \rightarrow \langle \Phi f, g \rangle \text{ for each } f, g \in L^2(X, \mathcal{B}, \mu).$$

By transferring the weak operator topology via (2) we obtain the weak topology on  $J(T)$  and

$$\eta_n \rightarrow \eta \text{ iff } \eta_n(A \times B) \rightarrow \eta(A \times B) \text{ for each } A, B \in \mathcal{B}.$$

Since the composition of two Markov operators is Markov,  $\mathcal{J}(T)$  is a compact semitopological semigroup. By the same token,  $J(T)$  is also a compact semitopological semigroup ( $\eta_1 \circ \eta_2 := \eta_{\Phi_{\eta_1} \circ \Phi_{\eta_2}}$ ).

Given a *factor*<sup>6</sup>, i.e. a  $T$ -invariant sub- $\sigma$ -algebra  $\mathcal{A} \subset \mathcal{B}$ , let

$$\mu = \int_{X/\mathcal{A}} \delta_{\bar{x}} \otimes \mu_{\bar{x}} d\mu(\bar{x})$$

be the disintegration of  $\mu$  over the factor  $\mathcal{A}$ . By setting

$$\mu \otimes_{\mathcal{A}} \mu = \int_{X/\mathcal{A}} \delta_{\bar{x}} \otimes \mu_{\bar{x}} \otimes \mu_{\bar{x}} d\mu(\bar{x}).$$

we obtain a self-joining  $\mu \otimes_{\mathcal{A}} \mu$  which is often called the *relative product over*  $\mathcal{A}$ . Note that  $\mu \otimes_{\mathcal{A}} \mu|_{\mathcal{A} \otimes \mathcal{A}} = \Delta_{\mathcal{A}}$ , where  $\Delta_{\mathcal{A}}(A_1 \times A_2) = \mu(A_1 \cap A_2)$  for each  $A_1, A_2 \in \mathcal{A}$ . Moreover, we have  $\Phi_{\mu \otimes_{\mathcal{A}} \mu} = E(\cdot | \mathcal{A})$ .

Assume additionally that  $T$  is ergodic. Then we can speak about ergodic self-joinings of  $T$  and the set of such joinings will be denoted by  $J^e(T)$ . By  $\mathcal{J}^e(T)$  we denote the subset of  $\mathcal{J}(T)$  corresponding to  $J^e(T)$ . The elements

---

<sup>6</sup>Up to a little abuse of notation, we define the *factor system*  $T|_{\mathcal{A}} : (X/\mathcal{A}, \mathcal{A}, \mu|_{\mathcal{A}}) \rightarrow (X/\mathcal{A}, \mathcal{A}, \mu|_{\mathcal{A}})$  in which cosets  $\bar{x} \in X/\mathcal{A}$  are given by those points which cannot be distinguished by the sets from  $\mathcal{A}$ ; then  $T|_{\mathcal{A}}(\bar{x}) = \overline{T x}$ .

of  $\mathcal{J}^e(T)$  are exactly the extremal points in the natural simplex structure of  $\mathcal{J}(T)$ . Recall that  $T$  is said to be *relatively weakly mixing* over a factor  $\mathcal{A}$  if  $E(\cdot|\mathcal{A}) \in \mathcal{J}^e(T)$ .

The notion which is complementary to relative weak mixing is the concept of relative Kronecker factor [8], [30]. More precisely, if  $\mathcal{A}$  is a factor then the *relative Kronecker factor*  $\mathcal{K}(\mathcal{A})$  (of  $T$  over  $T|_{\mathcal{A}}$ ) is the smallest  $\sigma$ -algebra making all relative eigenfunctions<sup>7</sup> measurable ( $\mathcal{A} \subset \mathcal{K}(\mathcal{A})$ ).

For more about joinings or relative concepts in ergodic theory, see e.g. [8], [12], [19], [28] and [30].

## 1.2 $G$ -actions

Assume that  $G$  is a second countable LCA group. By a  $G$ -action  $\mathcal{S} = (S_g)_{g \in G}$  we mean a *measurable representation* of  $G$  on a probability standard Borel space  $(Y, \mathcal{C}, \nu)$ , that is a group homomorphism  $g \mapsto S_g$ ,  $G \rightarrow \text{Aut}(Y, \mathcal{C}, \nu)$ . Then we also denote by  $\mathcal{S} = (S_g)_{g \in G}$  the associated unitary representation of  $G$  on  $L^2(Y, \mathcal{C}, \nu)$ , which is continuous. For each  $f \in L^2(Y, \mathcal{C}, \nu)$ , by  $\sigma_{f, \mathcal{S}}$  (or  $\sigma_f$  if  $\mathcal{S}$  is understood) we denote the *spectral measure* of  $f$ , *i.e.* the measure on the character group  $\widehat{G}$ <sup>8</sup> determined by the Fourier transform<sup>9</sup>

$$\widehat{\sigma}_{f, \mathcal{S}}(g) := \int_{\widehat{G}} \chi(g) d\sigma_{f, \mathcal{S}}(\chi) = \int_Y f \circ S_g \cdot \bar{f} d\nu.$$

We denote  $G(f) = \overline{\text{span}}\{S_g f : g \in G\}$ . Then the correspondence  $f \rightarrow 1_{\widehat{G}}$  yields the canonical isomorphism of  $\mathcal{S}|_{G(f)}$  with the representation  $\mathcal{V}_{\sigma_f} = (V_g^{\sigma_f})_{g \in G}$  of  $L^2(\widehat{G}, \mathcal{B}(\widehat{G}), \sigma_f)$ , where  $V_g^{\sigma_f} j(\chi) = \chi(g) j(\chi)$ . The maximal spectral type of  $\mathcal{S}$  on  $L_0^2(Y, \mathcal{C}, \nu)$  (the subspace of zero mean function in  $L^2(Y, \mathcal{C}, \nu)$ ) will be denoted by  $\sigma_{\mathcal{S}}$ <sup>10</sup>.

---

<sup>7</sup>By a *relative* (with respect to  $\mathcal{A}$ ) *eigenvalue* of  $T$  one means an  $\mathcal{A}$ -measurable map  $c : (X/\mathcal{A}, \mathcal{A}, \mu|_{\mathcal{A}}) \rightarrow U(n)$  for which there is  $M : (X, \mathcal{B}, \mu) \rightarrow \mathbb{C}^n$  satisfying the following:

$$c(\overline{x}) \begin{pmatrix} M_1(x) \\ \vdots \\ M_n(x) \end{pmatrix} = \begin{pmatrix} M_1(Tx) \\ \vdots \\ M_n(Tx) \end{pmatrix} \quad \text{for a.e. } x \in X, \quad (3)$$

$$M_i \perp_{\mathcal{A}} M_j \quad \text{for } i \neq j \quad \text{and } E(|M_i|^2 | \mathcal{A}) = 1, \quad i, j = 1, \dots, n. \quad (4)$$

The map  $M$  satisfying (3) and (4) is called a *relative eigenfunction corresponding to  $c$* .

<sup>8</sup>Since  $G$  is second countable LCA, also  $\widehat{G}$  is second countable LCA.

<sup>9</sup>By Pontryagin Duality Theorem, the character group of  $\widehat{G}$  has a natural identification with  $G$ .

<sup>10</sup>Formally speaking, it is the class of equivalence of measures which are maximal spectral measures but in what follows we abuse the vocabulary and often speak about a given

For more about the spectral theory of  $G$ -actions, see *e.g.* [19], [20].

Suppose that  $\mathcal{S}_i = (S_g^{(i)})_{g \in G}$  is a  $G$ -action on  $(Y_i, \mathcal{C}_i, \nu_i)$ ,  $i = 1, 2$ . By a *joining* of these two  $G$ -actions we mean an  $(S_g^{(1)} \times S_g^{(2)})_{g \in G}$ -invariant measure on  $(Y_1 \times Y_2, \mathcal{C}_1 \otimes \mathcal{C}_2)$  with projections  $\nu_1$  and  $\nu_2$  respectively<sup>11</sup>. Recall that  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are called *disjoint* (in the sense of Furstenberg [7]) if the only possible joining between them is product measure. We then write  $\mathcal{S}_1 \perp \mathcal{S}_2$ . It is well-known [14] that

$$\sigma_{\mathcal{S}_1} \perp \sigma_{\mathcal{S}_2} \Rightarrow \mathcal{S}_1 \perp \mathcal{S}_2.$$

Denote by  $M(\widehat{G})$  the convolution Banach algebra of all complex Borel measures on  $\widehat{G}$ <sup>12</sup>. Let  $M^+(\widehat{G}) \subset M(\widehat{G})$  (respectively  $M^{+,1}(\widehat{G}) \subset M(\widehat{G})$ ) consists of nonnegative members of  $M(\widehat{G})$  (of all probability measures in  $M(\widehat{G})$ ).

Assume that  $G$  is not compact. Recall that  $\sigma \in M^+(\widehat{G})$  is called *Dirichlet* if  $\limsup_{g \rightarrow \infty} |\widehat{\sigma}(g)| = \sigma(\widehat{G})$  (or equivalently if there exists a sequence  $g_n \rightarrow \infty$  in  $G$  such that  $\widehat{\sigma}(g_n) \rightarrow \sigma(\widehat{G})$ ).

### 1.3 $G$ -valued cocycles for an ergodic automorphism

Assume that  $T \in \text{Aut}(X, \mathcal{B}, \mu)$ . Let  $G$  be a second countable LCA group<sup>13</sup>. Let  $\varphi : X \rightarrow G$  be measurable. It determines a cocycle  $\varphi(n, x) = \varphi^{(n)}(x)$ <sup>14</sup> by the following formula

$$\varphi^{(n)}(x) = \begin{cases} \varphi(x) \cdot \varphi(Tx) \cdot \dots \cdot \varphi(T^{n-1}x) & \text{if } n > 0 \\ 1 & \text{if } n = 0 \\ (\varphi(T^n x) \cdot \dots \cdot \varphi(T^{-1}x))^{-1} & \text{if } n < 0. \end{cases}$$

Let us recall now the definitions of two groups related to  $T$  and  $\varphi$  that play basic role in the study of Rokhlin cocycle extensions (studied in Section 3).

**The group  $\Lambda_\varphi$ :** This is a Borel subgroup of  $\widehat{G}$  defined as

$$\Lambda_\varphi = \{\chi \in \widehat{G} : \chi \circ \varphi = \xi / \xi \circ T \text{ for a measurable } \xi : X \rightarrow \mathbb{T}\}^{15}.$$

---

measure as the maximal spectral type.

<sup>11</sup>Slightly generalizing Section 1.1,  $\eta$  determines a Markov intertwining operator  $\Phi_\eta : L^2(Y_1, \mathcal{C}_1, \nu_1) \rightarrow L^2(Y_2, \mathcal{C}_2, \nu_2)$ ; the correspondence similar to (2) also takes place.

<sup>12</sup>Since  $\widehat{G}$  is Polish, all members of  $M(\widehat{G})$  are regular measures.

<sup>13</sup>Here and all over the paper we use multiplicative notation.

<sup>14</sup> $\varphi(\cdot, \cdot)$  satisfies the cocycle identity  $\varphi(m+n, \cdot) = \varphi(m, \cdot) \cdot \varphi(n, T^m \cdot)$ ; it is often  $\varphi$  itself which is called a cocycle. A cocycle  $\varphi : X \rightarrow G$  is called a *coboundary* if  $\varphi = f/f \circ T$  for a measurable  $f : X \rightarrow G$ . If two cocycles differ by a coboundary then they are called *cohomologous*. A cocycle is said to be a *quasi-coboundary* if it is cohomologous to a constant cocycle.

<sup>15</sup>We denote  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ .

This group turns out to be the group of  $L^\infty$ -eigenvalues of the Mackey action (of  $G$ ) associated to the cocycle  $\varphi$  (see *e.g.* [1], [15], [18], [22]).

**The group  $\Sigma_\varphi$ :** This is a Borel subgroup of  $\widehat{G}$  defined as

$$\Sigma_\varphi = \{\chi \in \widehat{G} : \chi \circ \varphi = c \cdot \xi / \xi \circ T \text{ for a measurable } \xi : X \rightarrow \mathbb{T} \text{ and } c \in \mathbb{T}\}.$$

## 2 Tools

In this section we will present tools that will be needed to prove lifting of various properties by Rokhlin cocycles (see Section 3). Some of the results that will be presented here are new and seem to be of independent interest (see Section 2.2).

### 2.1 Idempotents in $J(T)$

Assume that  $T \in \text{Aut}(X, \mathcal{B}, \mu)$ . Then the closure of the group  $\{T^j : j \in \mathbb{Z}\}$  in  $\mathcal{J}(T)$ , denoted by  $\overline{\{T^j : j \in \mathbb{Z}\}}$ , is a closed subsemigroup of  $\mathcal{J}(T)$  and therefore

$$\overline{\{T^j : j \in \mathbb{Z}\}} \text{ is a semitopological compact semigroup.} \quad (5)$$

Given a factor  $\mathcal{A} \subset \mathcal{B}$ , we have  $E(\cdot|\mathcal{A}) = \Phi_{\mu \otimes_{\mathcal{A}} \mu}$ . Notice that given  $\Phi \in \mathcal{J}(T)$ ,

$$\Phi \circ E(\cdot|\mathcal{A}) = E(\cdot|\mathcal{A}) \text{ if and only if } \Phi f = f \text{ for each } f \in L^2(\mathcal{A}). \quad (6)$$

It follows that

$$\Phi \circ E(\cdot|\mathcal{A}) = E(\cdot|\mathcal{A}) \text{ if and only if } \eta_\Phi|_{\mathcal{A} \otimes \mathcal{A}} = \Delta_{\mathcal{A}}. \quad (7)$$

Indeed, assume  $\eta_\Phi|_{\mathcal{A} \otimes \mathcal{A}} = \Delta_{\mathcal{A}}$ , fix  $f \in L^2(\mathcal{A})$  and let  $g \in L^2(\mathcal{A})$  be arbitrary. Then

$$\int_X \Phi f(y) g(y) d\mu(y) = \int_{X \times X} f(x) g(y) d\eta_\Phi(x, y) = \int_X f g d\mu.$$

Since  $g$  was arbitrary in  $L^2(\mathcal{A})$ ,  $\Phi f - f$  is orthogonal to  $L^2(\mathcal{A})$ . But we must have  $\|\Phi f\| \leq \|f\|$ , and thus  $\Phi f = f$ . Conversely, if  $\Phi f = f$  for each  $f \in L^2(\mathcal{A})$ , we get the same equalities for all  $f, g \in L^2(\mathcal{A})$ , whence  $\eta_\Phi|_{\mathcal{A} \otimes \mathcal{A}} = \Delta_{\mathcal{A}}$ . Now (7) follows from (6).

In view of (7) we obtain that

$$\{\Phi \in \mathcal{J}(T) : \eta_\Phi|_{\mathcal{A} \otimes \mathcal{A}} = \Delta_{\mathcal{A}}\} \text{ is a compact semitopological semigroup.} \quad (8)$$

**Lemma 1** *Assume that  $\eta \in J(T)$ . Then*

$$\begin{aligned} L^2(\mathcal{B} \otimes \{\emptyset, X\}, \eta) \cap L^2(\{\emptyset, X\} \otimes \mathcal{B}, \eta) \\ = \{f \otimes 1 : f \in L^2(X, \mathcal{B}, \mu), \|\Phi_\eta f\| = \|f\|\}. \end{aligned} \quad (9)$$

**Proof.**

Suppose that  $f(x) = g(y)$  for  $\eta$ -a.e.  $(x, y) \in X \times X$ . Then we have  $\Phi_\eta f = g$  and

$$\|f\|^2 = \int_{X \times X} |f(x)|^2 d\eta(x, y) = \int_{X \times X} |g(y)|^2 d\eta(x, y) = \|g\|^2,$$

so  $\|\Phi_\eta f\| = \|f\|$ .

On the other hand, take  $f \in L^2(X, \mathcal{B}, \mu)$  satisfying  $\|\Phi_\eta f\| = \|f\|$ . Then similarly

$$\int_{X \times X} |f(x)|^2 d\eta(x, y) = \int_{X \times X} |\Phi_\eta f(y)|^2 d\eta(x, y).$$

But, immediately from the definition, the function  $(x, y) \mapsto \Phi_\eta f(y)$  is the orthogonal projection of  $(x, y) \mapsto f(x)$  on the subspace  $L^2(\{\emptyset, X\} \otimes \mathcal{B})$  of  $L^2(X \times X, \mathcal{B} \otimes \mathcal{B}, \eta)$ . It follows that  $f(x) = \Phi_\eta f(y)$   $\eta$ -a.e.  $(x, y)$ .  $\square$

The  $\sigma$ -algebra  $\mathcal{B} \otimes \{\emptyset, X\} \cap \{X, \emptyset\} \otimes \mathcal{B}$  (modulo  $\eta$ ) can be seen on one hand as a factor  $\mathcal{B}_1(\eta) \otimes \{\emptyset, X\}$  of  $\mathcal{B} \otimes \{\emptyset, X\}$  and on the other hand as a factor  $\{\emptyset, X\} \otimes \mathcal{B}_2(\eta)$  of  $\{\emptyset, X\} \otimes \mathcal{B}$ . This defines two factors  $\mathcal{B}_1(\eta)$ ,  $\mathcal{B}_2(\eta)$  of  $(X, \mathcal{B}, \mu)$ , the largest factors identified by the joining  $\eta$ .

Whenever  $\mathcal{A} \subset \mathcal{B}$  is a factor of  $T$ , the relative product  $\mu \otimes_{\mathcal{A}} \mu$  is an idempotent in  $J(T)$ . The following result states that this is the only way to obtain idempotents in  $J(T)$  (cf. Theorem 6.9 in [12] where self-adjoint idempotents of  $\mathcal{J}(T)$  are shown to correspond to factors).

**Proposition 1** *Assume that  $\eta$  is an idempotent in  $J(T)$ . Then there exists a factor  $\mathcal{A}$  of  $T$  such that  $\eta = \mu \otimes_{\mathcal{A}} \mu$ .*

**Proof.**

Since  $\|\Phi_\eta\| = 1$ , it must be an orthogonal projection and it is an isometry exactly on its range. Now, in view of Lemma 1,  $\Phi_\eta$  is an isometry exactly on  $L^2(\mathcal{B}_1(\eta))$ . Therefore it is the orthogonal projection onto  $L^2(\mathcal{B}_1(\eta))$ , that is  $\Phi_\eta = E(\cdot | \mathcal{B}_1(\eta))$  and the result follows.  $\square$



We can see factors of the form  $\mathcal{B}_1(\eta)$  in a different way. Indeed, given  $\eta \in J(T)$  define

$$\mathcal{B}(\eta) := \{A \in \mathcal{B} : \eta((A \times X) \triangle (X \times A)) = 0\}.$$

Then  $L^2(\mathcal{B}(\eta)) = \{f \in L^2(X, \mathcal{B}, \mu) : \Phi_\eta f = f\}$ . Indeed, from the von Neumann theorem for contractions,  $\frac{1}{N} \sum_{n=0}^{N-1} \Phi_\eta^n \rightarrow \text{proj}_{\text{Fix}(\Phi_\eta)}$  and since the limit is an idempotent and a Markov operator, it is the orthogonal projection on the  $L^2$ -space of a factor.

Recall also that if  $W$  is a contraction of a Hilbert space  $H$  then so is its adjoint and then  $W^*Wf = f$  if and only if  $\|Wf\| = \|f\|$ . Hence for any  $\eta \in J(T)$ ,

$$\mathcal{B}_1(\eta) = \mathcal{B}(\eta^* \circ \eta),$$

where  $\eta^* := \eta_{\Phi^*}$ .

An automorphism  $T$  is said to be *rigid* if there exists a sequence  $q_n \rightarrow \infty$  such that  $T^{q_n} \rightarrow \text{Id}$  in the strong (or, which here is the same, in the weak) operator topology. We then say that  $(q_n)$  is a *rigidity* sequence for  $T$ . Suppose now that  $\mathcal{A}$  is a non-trivial factor of  $T$  and suppose moreover that  $(q_n)$  is a rigidity sequence for  $T|_{\mathcal{A}}$ . Consider

$$I(\mathcal{A}) := \{\Phi \in \mathcal{J}(T) : \eta_\Phi|_{\mathcal{A} \otimes \mathcal{A}} = \Delta_{\mathcal{A}} \text{ and } \Phi \text{ is a limit point of } \{T^j : j \in \mathbb{Z}\}\}. \quad (10)$$

Note that  $I(\mathcal{A})$  is non-empty since any limit Markov operator of the set  $\{T^{q_n} : n \geq 1\}$  belongs to it. It follows from (7) and (8) that  $I(\mathcal{A})$  is a closed subsemigroup of  $\mathcal{J}(T)$ , hence a semitopological compact semigroup. We recall (see *e.g.* [10], p. 6, Lemma 2.2) that each compact semitopological semigroup contains an idempotent. Now, using Proposition 1, we obtain the following.

**Proposition 2** *Assume that  $T$  is an automorphism of  $(X, \mathcal{B}, \mu)$  and let  $\mathcal{A} \subset \mathcal{B}$  be a non-trivial rigid factor of  $T$ . Then there exist a factor  $\mathcal{A}'$  containing  $\mathcal{A}$  and a rigidity sequence  $(q_n)$  for  $T|_{\mathcal{A}'}$  such that  $T^{q_n} \rightarrow E(\cdot|_{\mathcal{A}'}).$*   $\square$

## 2.2 Canonical factor of a $G$ -action associated to a saturated Borel subgroup

Assume that  $\Lambda$  is a Borel subgroup of  $\widehat{G}$ . Let us recall (see [17]) that if  $\sigma, \tau \in M^{+,1}(\widehat{G})$  then  $\tau$  *sticks to*  $\sigma$  if

$$\widehat{\sigma}(g_j) \rightarrow 1 \implies \widehat{\tau}(g_j) \rightarrow 1$$

for any sequence  $(g_j)_{j \geq 1}$  in  $G$  going to infinity. Following [17], one says that  $\Lambda$  is *saturated* if for any  $\sigma, \tau \in M^{+,1}(\widehat{G})$

$$\sigma(\Lambda) = 1 \text{ and } \tau \text{ sticks to } \sigma \implies \tau(\Lambda) = 1.$$

**Theorem 1** ([17]) *Every group  $\Lambda_\varphi$  is saturated.*  $\square$

**Remark 1** As noticed *e.g.* in [22], every subgroup  $\Sigma_\varphi$  is also of the form  $\Lambda_\psi$ , whence  $\Sigma_\varphi$  is also a saturated subgroup.

We shall also need the following characterization of saturated groups.

**Theorem 2** ([17]) *A Borel subgroup  $\Lambda \subset \widehat{G}$  is saturated if and only if for any  $\tau \in M^+(\widehat{G})$  the indicator function  $1_\Lambda$  belongs to the closed convex hull in  $L^1(\widehat{G}, \tau)$  of the characters of  $\widehat{G}$ .*  $\square$

The following corollary describes a dynamical consequence of Theorem 2<sup>16</sup>. Given a Borel subset  $\Lambda$  of  $\widehat{G}$ , we denote by  $H_\Lambda$  the spectral subspace corresponding to  $\Lambda$ , *i.e.* the space of those elements in  $L^2(Y, \mathcal{C}, \nu)$  whose spectral measures are concentrated on  $\Lambda$ . We denote by  $\tilde{g}$  the character of  $\widehat{G}$  associated by Pontryagin duality to  $g \in G$ :  $\tilde{g}(\chi) := \chi(g)$  for  $\chi \in \widehat{G}$ .

**Corollary 1** *Let  $\mathcal{S}$  be an action of  $G$  on  $(Y, \mathcal{C}, \nu)$  and  $\Lambda$  be a saturated subgroup of  $\widehat{G}$ . Then  $H_\Lambda = L^2(\mathcal{A})$  where  $\mathcal{A} \subset \mathcal{C}$  is a factor of  $\mathcal{S}$ .*

**Proof.**

First notice that in Theorem 2,  $L^1$ -convergence can be replaced by  $L^2$ -convergence, in particular

$$\sum_{k=1}^{N_n} a_k^{(n)} \tilde{g}_k^{(n)}(\chi) \rightarrow 1_\Lambda \text{ in } L^2(\widehat{G}, \mathcal{B}(\widehat{G}), \sigma_{\mathcal{S}})$$

for some  $a_k^{(n)} \geq 0$  with  $\sum_{k=1}^{N_k} a_k^{(n)} = 1$  and some  $g_k^{(n)} \in G$ . Then, for each  $\tau \in M^+(\widehat{G})$ ,  $\tau \ll \sigma_{\mathcal{S}}$  we still have

$$\sum_{k=1}^{N_n} a_k^{(n)} \tilde{g}_k^{(n)}(\chi) \rightarrow 1_\Lambda \text{ in } L^2(\widehat{G}, \mathcal{B}(\widehat{G}), \tau). \quad (11)$$

---

<sup>16</sup>This consequence of Theorem 2 seems to appear for the first time.

Consider  $f \in L^2(Y, \mathcal{C}, \nu)$ . In the canonical representation of  $G(f)$ , the function  $\sum_{k=1}^{N_n} a_k^{(n)} \tilde{g}_k^{(n)} \in L^2(\widehat{G}, \mathcal{B}(\widehat{G}), \sigma_f)$  corresponds to  $\sum_{k=1}^{N_n} a_k^{(n)} S_{g_k^{(n)}} f$  and the subspace  $1_\Lambda \cdot L^2(\widehat{G}, \mathcal{B}(\widehat{G}), \sigma_f)$  corresponds to  $H_\Lambda \cap G(f)$ . So, by taking  $\tau = \sigma_f$  in (11),

$$\sum_{k=1}^{N_n} a_k^{(n)} S_{g_k^{(n)}} f \rightarrow \text{proj}_{H_\Lambda \cap G(f)} f.$$

Now, since  $H_\Lambda$  is a spectral subspace,  $\text{proj}_{H_\Lambda} f = \text{proj}_{H_\Lambda \cap G(f)} f$ . It follows that the sequence  $\left( \sum_{k=1}^{N_n} a_k^{(n)} S_{g_k^{(n)}} \right)_{n \geq 1}$  of Markov operators of  $L^2(Y, \mathcal{C}, \nu)$  converges weakly to  $\text{proj}_{H_\Lambda}$ . Therefore, the latter projection is a Markov operator and the result follows from Proposition 1.  $\square$

The following lemma allows us to localize some eigenvalues of  $\mathcal{S}$ .

**Lemma 2** *Let  $\mathcal{S}$  be a  $G$ -action on  $(Y, \mathcal{C}, \nu)$  and  $\Lambda$  be a saturated subgroup of  $\widehat{G}$ . Assume that  $\sigma_{\mathcal{S}}(\Lambda) = 0$  and that  $\sigma_{\mathcal{S}}(\chi_0 \Lambda) > 0$  for some  $\chi_0 \in \widehat{G}$ . Then  $\mathcal{S}$  has an eigenvalue in  $\chi_0 \Lambda$ . More precisely, there exists exactly one eigenvalue of  $\mathcal{S}$  in  $\chi_0 \Lambda$  and  $H_{\chi_0 \Lambda}$  is the eigenspace corresponding to that eigenvalue.*

**Proof.**

Denote by  $\Gamma$  the cyclic group  $\{\chi_0^n : n \in \mathbb{Z}\}$  considered with the discrete topology. Let  $Z$  be the dual group of  $\Gamma$ . Hence we obtain a probability space  $(Z, \mathcal{D}, \eta)$ , where  $\mathcal{D} = \mathcal{B}(Z)$  and  $\eta$  is the normalized Haar measure on  $Z$ . Given  $g \in G$  we define  $\tilde{g} \in Z$  by  $\tilde{g}(\chi) = \chi(g)$  for each  $\chi \in \Gamma$ , and  $R_g : Z \rightarrow Z$  by  $R_g(z) = \tilde{g} \cdot z$ . In this way we obtain an ergodic discrete spectrum  $G$ -action  $\mathcal{R} = (R_g)_{g \in G}$  on  $(Z, \mathcal{D}, \eta)$ , whose point spectrum is equal to  $\Gamma$ . Let us consider the diagonal  $G$ -action  $\mathcal{S} \times \mathcal{R} = (S_g \times R_g)_{g \in G}$  on  $(Y \times Z, \mathcal{C} \otimes \mathcal{D}, \nu \otimes \eta)$ . The maximal spectral type  $\sigma_{\mathcal{S} \times \mathcal{R}}$  of the associated unitary  $G$ -action is equal to

$$\left( \sum_{n \in \mathbb{Z}} \frac{1}{2^{|n|}} \delta_{\chi_0^n} \right) * \sigma_{\mathcal{S}} + \sum_{n \neq 0} \frac{1}{2^{|n|}} \delta_{\chi_0^n}.$$

By the assumption, we have  $\sigma_{\mathcal{S} \times \mathcal{R}}(\Lambda) > 0$ . In view of Corollary 1, there exists a non-trivial  $\mathcal{S} \times \mathcal{R}$ -factor  $\mathcal{A} \subset \mathcal{C} \otimes \mathcal{D}$  for which

$$L^2(\mathcal{A}) = \{F \in L^2(Y \times Z, \nu \otimes \eta) : \sigma_{F, \mathcal{S} \times \mathcal{R}} \text{ is concentrated on } \Lambda\}.$$

Now fix a non-zero  $f \in H_{\chi_0\Lambda}$  and let  $h$  be the eigenfunction  $z \mapsto \overline{z(\chi_0)}$  of  $\mathcal{R}$ , corresponding to the eigenvalue  $\overline{\chi_0}$ . Then  $\sigma_{f \otimes h, \mathcal{S} \times \mathcal{R}} = \delta_{\overline{\chi_0}} * \sigma_{f, \mathcal{S}}$ , hence

$$\sigma_{f \otimes h, \mathcal{S} \times \mathcal{R}} \text{ is concentrated on } \Lambda \quad (12)$$

and  $f \otimes h \in L_0^2(\mathcal{A})$ . Consider the function  $|f|^2 \otimes h$ . First notice that

$$|f|^2 \otimes h = (f \otimes h) \cdot (\overline{f} \otimes 1),$$

so the function  $|f|^2 \otimes h$  is measurable with respect to  $\mathcal{A} \vee (\mathcal{C} \otimes \{\emptyset, Z\})$ .

The two  $G$ -actions  $(\mathcal{S} \times \mathcal{R})|_{\mathcal{A}}$  and  $\mathcal{S}$  are spectrally disjoint since, by assumption,  $\sigma_{\mathcal{S}}(\Lambda) = 0$ . Hence, they are disjoint. In particular,  $|f|^2 \otimes h$  is in  $L^2(Y \times Z, \nu \otimes \eta)$  and

$$\sigma_{|f|^2 \otimes h, \mathcal{S} \times \mathcal{R}} = \sigma_{f \otimes h, (\mathcal{S} \times \mathcal{R})|_{\mathcal{A}}} * \sigma_{\overline{f}, \mathcal{S}}. \quad (13)$$

At the same time, since  $\int_Y |f|^2 d\nu > 0$ , we have  $\delta_1 \ll \sigma_{|f|^2, \mathcal{S}}$  and therefore

$$\sigma_{|f|^2 \otimes h, \mathcal{S} \times \mathcal{R}} = \sigma_{|f|^2, \mathcal{S}} * \sigma_{h, \mathcal{R}} \gg \delta_1 * \delta_{\overline{\chi_0}} = \delta_{\overline{\chi_0}}.$$

It now follows directly from (13) that  $\sigma_{\overline{f}, \mathcal{S}}$  is not a continuous measure. More precisely, in view of (12),  $\sigma_{\overline{f}, \mathcal{S}}$  must have a point mass at some  $\chi \in \widehat{G}$  such that  $\overline{\chi_0} \in \chi\Lambda$ , and  $f$  cannot be orthogonal to the subspace of eigenfunctions corresponding to the eigenvalues of  $\mathcal{S}$  in  $\chi_0\Lambda$ . Since  $f$  is an arbitrary element of  $H_{\chi_0\Lambda}$ , the space  $H_{\chi_0\Lambda}$  consists only of eigenfunctions. Finally, since  $\sigma_{\mathcal{S}}(\Lambda) = 0$ , no two different eigenvalues of  $\mathcal{S}$  can be in the coset  $\chi_0\Lambda$  and the proof is complete.  $\square$

**Remark 2** Note that  $\sigma_{\mathcal{S}}(\Lambda) = 0$  implies that  $\mathcal{S}$  is ergodic. It follows that under the assumptions of the above lemma,  $H_{\chi_0\Lambda}$  is moreover one-dimensional.

Although, the result below (Proposition 3) will not be used in what follows, we bring it up, as it is another sample of applications of saturated groups in (non-singular) ergodic theory.

Assume that  $T$  is a non-singular ergodic automorphism of a standard probability space  $(X, \mathcal{B}, \mu)$  and that  $S : (Y, \mathcal{C}, \nu) \rightarrow (Y, \mathcal{C}, \nu)$  is another non-singular automorphism. Let  $\pi : (X, \mathcal{B}, \mu) \rightarrow (Y, \mathcal{C}, \nu)$  settle a homomorphism between  $T$  and  $S$ . Following [5],  $(S, \pi)$  is called a *relatively finite measure-preserving* (rfmp) factor of  $T$  if  $\frac{d\mu \circ T}{d\mu}$  is  $\pi^{-1}(\mathcal{C})$ -measurable.

**Proposition 3** *Assume that  $T$  is a nonsingular ergodic automorphism and  $S$  is an rfmp factor of it. Let  $R$  be a weakly mixing probability preserving automorphism of a standard probability Borel space  $(Z, \mathcal{D}, \eta)$ . Assume that  $R \times S$  is ergodic. Then  $R \times T$  is also ergodic*<sup>17</sup>.

**Proof.**

We need to show that  $\sigma_R(e(T)) = 0$ , where  $e(T)$  stands for the group of  $L^\infty$  eigenvalues ( $c \in e(T) \subset \mathbb{T}$  if for some  $f \in L^\infty(X, \mathcal{B}, \mu)$ ,  $f \circ T = c \cdot f$ ), see e.g. [23]. In view of Theorem 2 in [4],  $e(T)$  is the union of countably many cosets  $c \cdot e(S)$ ,  $e(T) = \bigcup_{i=1}^\infty c_i \cdot e(S)$ . On the other hand,  $\sigma_R(e(S)) = 0$  and  $e(S)$  is saturated (see [17]). Since  $R$  is weakly mixing, in view of Lemma 2,  $\sigma_R(c \cdot e(S)) = 0$  for each  $c \in \mathbb{T}$ . Therefore,  $\sigma_R(e(T)) = 0$  and the result follows.  $\square$

Following [5], given a non-singular automorphism  $S$  of  $(Y, \mathcal{C}, \nu)$ , to obtain  $T$  and  $\pi$  so that  $(S, \pi)$  is an rfmp factor of  $T$  we must take any ergodic skew product  $T = S_\Theta$  on  $(X, \mathcal{B}, \mu) = (Y, \mathcal{C}, \nu) \otimes (X', \mathcal{B}', \mu')$  where  $(X', \mathcal{B}', \mu')$  is another probability standard Borel space and  $\Theta : X \rightarrow \text{Aut}(X', \mathcal{B}', \mu')$  is measurable.

### 2.3 Lemma on mixing times of weighted unitary operators

In the study of mixing properties of automorphism of the form  $T_{\varphi, S}$  unitary operators  $V_\xi$  defined below will play a crucial role.

Let  $T$  be an automorphism of  $(X, \mathcal{B}, \mu)$ . Let  $\xi : X \rightarrow \mathbb{T}$  be a cocycle. We define a unitary operator  $V_\xi$  on  $L^2(X, \mathcal{B}, \mu)$  by setting

$$V_\xi(f)(x) = \xi(x) \cdot f(Tx)$$

for each  $f \in L^2(X, \mathcal{B}, \mu)$ . A sequence  $(n_i) \subset \mathbb{N}$ ,  $n_i \rightarrow \infty$  is said to be a *mixing sequence* for  $V_\xi$  if  $V_\xi^{n_i} \rightarrow 0$  in the weak operator topology (while  $(n_i)$  is a mixing sequence for  $T$  if  $T^{n_i}$  restricted to  $L_0^2(X, \mathcal{B}, \mu)$  goes to 0, that is,  $T^{n_i} \rightarrow \Phi_{\mu \otimes \mu}$  in  $\mathcal{J}(T)$ ).

Denote by  $T_\xi$  the Anzai skew product corresponding to  $T$  and  $\xi$ , i.e. the automorphism of  $(X \times \mathbb{T}, \mathcal{B} \otimes \mathcal{B}(\mathbb{T}), \mu \otimes \lambda)$  given by  $T_\xi(x, z) = (Tx, \xi(x)z)$ , where  $\lambda$  stands for the Lebesgue measure of the circle.

We assume now that  $T$  is ergodic.

---

<sup>17</sup>Usually, to ensure ergodicity one has to assume mild mixing of  $R$ , a property stronger than weak mixing. As a matter of fact,  $R$  is mildly mixing if and only if  $R \times T$  is ergodic for each non-singular ergodic  $T$ , see [9]. By definition ([9]), mild mixing means that  $R$  has no non-trivial rigid factors.

**Proposition 4** Assume that  $T^{n_i} \rightarrow \Phi$  in  $\mathcal{J}(T)$ , where  $\Phi = \Phi_\rho \in \mathcal{J}^e(T)$ . Suppose that the  $(T \times T, \rho)$ -cocycle  $\xi \otimes \bar{\xi}$  is not a coboundary. Then  $(n_i)$  is a mixing sequence for  $V_\xi$ .

**Proof.**

We can moreover assume that  $(T_\xi)^{n_i} \rightarrow \tilde{\Phi} = \Phi_{\tilde{\rho}}$  in  $\mathcal{J}(T_\xi)$ . Given  $f_1, f_2 \in L^2(X, \mathcal{B}, \mu)$ , we define  $F_i \in L^2(X \times \mathbb{T}, \mu \otimes \lambda)$  by  $F_i(x, z) = f_i(x)z$ ,  $i = 1, 2$ . Set also  $J(x_1, z_1, x_2, z_2) = z_1 \bar{z}_2$ , so  $J \in L^2((X \times \mathbb{T}) \times (X \times \mathbb{T}), \tilde{\rho})$ . Moreover let, with some abuse of notation,  $H$  denote  $E^{\tilde{\rho}}(J|\mathcal{B} \otimes \mathcal{B})$ . We have

$$\begin{aligned} \int_X (V_\xi)^{n_i} f_1 \cdot \bar{f}_2 d\mu &= \int_X \xi^{(n_i)} \cdot f_1 \circ T^{n_i} \cdot \bar{f}_2 d\mu \\ &= \int_{X \times \mathbb{T}} F_1 \circ T_\xi^{n_i} \cdot \bar{F}_2 d\mu d\lambda \\ &\longrightarrow \int_{(X \times \mathbb{T}) \times (X \times \mathbb{T})} F_1(x_1, z_1) \overline{F_2(x_2, z_2)} d\tilde{\rho}((x_1, z_1), (x_2, z_2)) \\ &= \int_{(X \times \mathbb{T}) \times (X \times \mathbb{T})} f_1(x_1) f_2(x_2) z_1 \bar{z}_2 d\tilde{\rho}((x_1, z_1), (x_2, z_2)) \\ &= \int_{X \times X} f_1 \otimes f_2 \cdot H d\rho. \end{aligned}$$

We claim now that, for  $\rho$ -a.a.  $(x_1, x_2) \in X \times X$ ,

$$\xi(x_1) \overline{\xi(x_2)} H(x_1, x_2) = H(Tx_1, Tx_2). \quad (14)$$

Indeed,  $J \circ (T_\xi \times T_\xi) = (\xi \otimes \bar{\xi}) \cdot J$ , so

$$\begin{aligned} E^{\tilde{\rho}}(J|\mathcal{B} \otimes \mathcal{B}) \circ (T \times T) &= E^{\tilde{\rho}}(J \circ (T_\xi \times T_\xi)|\mathcal{B} \otimes \mathcal{B}) \\ &= \xi \otimes \bar{\xi} \cdot E^{\tilde{\rho}}(J|\mathcal{B} \otimes \mathcal{B}) \end{aligned}$$

and (14) follows.

Now, ergodicity of  $\rho$  implies that  $H$  is of constant modulus. If  $H \neq 0$  then from (14) it follows that  $\xi \otimes \bar{\xi}$  is a  $(T \times T, \rho)$ -coboundary. Otherwise  $\int_X (V_\xi)^{n_i} f_1 \cdot \bar{f}_2 d\mu \rightarrow 0$  for all  $f_1, f_2 \in L^2(X, \mathcal{B}, \mu)$ , so  $(n_i)$  is a mixing sequence for  $V_\xi$ .  $\square$

**Corollary 2** If  $T$  is weakly mixing and  $(n_i)$  is a mixing sequence for  $T$ , then  $(n_i)$  is also a mixing sequence for  $V_\xi$  whenever  $\xi$  is not a quasi-coboundary.

**Proof.**

We have then  $T^{n_i} \rightarrow \Phi_{\mu \otimes \mu}$  in  $\mathcal{J}(T)$ . As  $T$  is weakly mixing,  $\mu \otimes \mu \in \mathcal{J}^e(T)$ , and it is well-known that  $\xi \otimes \bar{\xi}$  is a  $(T \times T, \mu \otimes \mu)$ -coboundary if and only if  $\xi$  is a quasi-coboundary (see *e.g.* [22], Appendix).  $\square$

## 2.4 Recurrent cocycles with values in Abelian Polish groups

The remarks below about recurrent cocycles with values in Polish Abelian groups are taken directly from the theory of cocycles taking values in LCA groups [29], we give proofs only for sake of completeness.

Let  $T$  be an ergodic automorphism of  $(X, \mathcal{B}, \mu)$ . Assume that  $A$  is an Abelian<sup>18</sup> Polish group. Let  $\varphi : X \rightarrow A$  be a cocycle. The cocycle  $\varphi$  is said to be *recurrent* if for each  $\varepsilon > 0$ ,  $B \in \mathcal{B}$  of positive measure and each neighbourhood  $V$  of 1, there exists a positive integer  $N$  such that

$$\mu(B \cap T^{-N}B \cap [\varphi^{(N)} \in V]) > 0. \quad (15)$$

Suppose that  $\psi : X \rightarrow A$  is another cocycle. Then we have the following fact:

$$\text{if } \varphi \text{ and } \psi \text{ are cohomologous and } \varphi \text{ is recurrent, then so is } \psi. \quad (16)$$

Indeed, assume that  $\psi = \varphi \cdot f \cdot (f \circ T)^{-1}$  for a measurable  $f : X \rightarrow A$ . Take a set  $B \in \mathcal{B}$  of positive measure. Fix a neighbourhood  $V$  of 1, and then another neighbourhood  $W$  of 1 so that  $W \cdot W \subset V$ . Using measurability of  $f$ , we can find a measurable subset  $B_1 \subset B$  of positive measure such that  $f(x) \cdot f(y)^{-1} \in W$  whenever  $x, y \in B_1$ . Then, for every  $N \geq 1$ ,

$$B_1 \cap T^{-N}B_1 \cap [\varphi^{(N)} \in W] \subset B \cap T^{-N}B \cap [\psi^{(N)} \in V]$$

and (16) follows.

Assume now that  $a \in A$  and let  $\tilde{a}$  denote the corresponding constant cocycle:  $\tilde{a}(x) = a$ . Then the following fact holds:

$$\begin{aligned} &\tilde{a} \text{ is recurrent if and only if} \\ &\text{there exists } n_j \rightarrow \infty \text{ such that } a^{n_j} \rightarrow 1 \text{ in } A. \end{aligned} \quad (17)$$

Indeed, fix a neighbourhood  $V$  of 1 and apply (15) with  $B = X$  to obtain that  $\mu([\tilde{a}^{(N)} \in V]) > 0$  for some positive integer  $N$ . It follows that  $a^N \in V$ .

---

<sup>18</sup>We keep going to use multiplicative notation.

Letting  $V \rightarrow \{1\}$ , either  $N = N(a, V) \rightarrow \infty$  and we are done, or  $N$  stays bounded. In the latter case we have  $a^N = 1$  for some  $N \geq 1$  and by taking multiples of this  $N$ , (17) also follows.

On the other hand, if  $a^{n_j} \rightarrow 1$  then the sequence of differences  $n_i - n_j$  is a sequence universally good for the Poincaré recurrence and clearly  $a^{n_i - n_j} \rightarrow 1$  when  $i, j \rightarrow \infty$ . Therefore  $\tilde{a}$  is recurrent.

### 3 Lifting mixing properties to Rokhlin cocycle extensions

In this section we will present a systematic study of mixing properties of automorphisms of the form  $T_{\varphi, \mathcal{S}}$ . Throughout  $T$  is assumed to be an automorphism of  $(X, \mathcal{B}, \mu)$ ,  $\varphi : X \rightarrow G$  a cocycle and  $\mathcal{S} = (S_g)_{g \in G}$  a  $G$ -action acting on  $(Y, \mathcal{C}, \nu)$ .

#### 3.1 Maximal spectral type of $T_{\varphi, \mathcal{S}}$

Let  $\{f_n\}_{n \geq 0}$  and  $\{g_n\}_{n \geq 0}$  be orthonormal bases in  $L^2(X, \mathcal{B}, \mu)$  and  $L^2(Y, \mathcal{C}, \nu)$  respectively, where  $f_0 = g_0 = 1$ . For the maximal spectral type  $\sigma_{T_{\varphi, \mathcal{S}}}$  of  $T_{\varphi, \mathcal{S}}$  on  $L^2_0(X \times Y, \mu \otimes \nu)$ , we take<sup>19</sup>

$$\sigma_{T_{\varphi, \mathcal{S}}} = \sum_{(m, n) \neq (0, 0)} 2^{-(m+n)} \sigma_{f_n \otimes g_m, T_{\varphi, \mathcal{S}}}. \quad (18)$$

According to the notation of section 2.3, given  $\chi \in \hat{G}$ , we denote by  $V_{\chi \circ \varphi}$  the unitary operator on  $L^2(X, \mathcal{B}, \mu)$  which acts by the formula

$$(V_{\chi \circ \varphi} f)(x) = \chi(\varphi(x)) f(Tx).$$

Its maximal spectral type, on  $L^2(X, \mathcal{B}, \mu)$ , is equal to

$$\sigma_{V_{\chi \circ \varphi}} = \sum_{n \geq 0} \frac{1}{2^n} \sigma_{f_n, V_{\chi \circ \varphi}}.$$

Notice also that the maximal spectral type of  $\mathcal{S}$  on  $L^2_0(Y, \mathcal{C}, \nu)$  is given by

$$\sigma_{\mathcal{S}} = \sum_{m \geq 1} \frac{1}{2^m} \sigma_{g_m, \mathcal{S}}$$

and  $\sigma_T$ , the maximal spectral type of  $T$  on  $L^2_0(X, \mathcal{B}, \mu)$ , is equal to  $\sum_{n \geq 1} \frac{1}{2^n} \sigma_{f_n, T}$ .

---

<sup>19</sup>Up to some abuse of vocabulary, we take as  $\sigma_{T_{\varphi, \mathcal{S}}}$  any spectral measure realizing the maximal spectral type.



**Lemma 3** *We have  $\sigma_{T_{\varphi,S}} = \sigma_T + \int_{\widehat{G}} \sigma_{V_{\chi \circ \varphi}} d\sigma_S(\chi)$ . Moreover*

$$\sigma_{T_{\varphi,S}}|_{L^2(X \times Y, \mu \otimes \nu) \ominus L^2(X, \mu) \otimes 1_Y} = \int_{\widehat{G}} \sigma_{V_{\chi \circ \varphi}} d\sigma_S(\chi).$$

**Proof.**

Firstly, we calculate the spectral measure of  $f \otimes g$  for  $f \in L^2(X, \mathcal{B}, \mu)$ ,  $g \in L^2(Y, \mathcal{C}, \nu)$ . For each  $k \in \mathbb{Z}$  we have

$$\begin{aligned} \widehat{\sigma}_{f \otimes g, T_{\varphi,S}}(k) &= \int_{X \times Y} (f \otimes g) \circ (T_{\varphi,S})^k \cdot \overline{f \otimes g} d(\mu \otimes \nu) \\ &= \int_X f(T^k x) \overline{f(x)} \left( \int_Y g(S_{\varphi^{(k)}(x)}(y)) \overline{g(y)} d\nu(y) \right) d\mu(x) \\ &= \int_X f(T^k x) \overline{f(x)} \left( \int_{\widehat{G}} \chi(\varphi^{(k)}(x)) d\sigma_{g,S}(\chi) \right) d\mu(x) \\ &= \int_{\widehat{G}} \left( \int_X \chi(\varphi^{(k)}(x)) f(T^k x) \overline{f(x)} d\mu(x) \right) d\sigma_{g,S}(\chi). \\ &= \int_{\widehat{G}} \widehat{\sigma}_{f, V_{\chi \circ \varphi}}(k) d\sigma_{g,S}(\chi). \end{aligned}$$

It follows that

$$\sigma_{f \otimes g, T_{\varphi,S}} = \int_{\widehat{G}} \sigma_{f, V_{\chi \circ \varphi}} d\sigma_{g,S}(\chi). \quad (19)$$

Therefore, in view of (18) and (19)

$$\begin{aligned} \sigma_{T_{\varphi,S}} &= \sum_{(n,m) \neq (0,0)} \frac{1}{2^{n+m}} \sigma_{f_n \otimes g_m, T_{\varphi,S}} \\ &= \int_{\widehat{G}} \sum_{(n,m) \neq (0,0)} \frac{1}{2^{n+m}} \sigma_{f_n, V_{\chi \circ \varphi}} d\sigma_{g_m,S}(\chi) \\ &= \sum_{m \geq 1} \frac{1}{2^m} \int_{\widehat{G}} \sum_{n \geq 0} \frac{1}{2^n} \sigma_{f_n, V_{\chi \circ \varphi}} d\sigma_{g_m,S}(\chi) + \int_{\widehat{G}} \sum_{n \geq 1} \frac{1}{2^n} \sigma_{f_n, V_{\chi \circ \varphi}} d\sigma_{g_0,S}(\chi) \\ &= \sum_{m \geq 1} \frac{1}{2^m} \int_{\widehat{G}} \sigma_{V_{\chi \circ \varphi}} d\sigma_{g_m,S}(\chi) + \sum_{n \geq 1} \frac{1}{2^n} \sigma_{f_n, T} \\ &= \int_{\widehat{G}} \sigma_{V_{\chi \circ \varphi}} d\sigma_S(\chi) + \sigma_T. \end{aligned}$$

The result immediately follows.  $\square$

### 3.2 Maximal spectral type of $T_{\varphi,S}$ on subspaces of the form $L^2(X, \mathcal{B}, \mu) \otimes G(g)$

Assume now that  $g \in L_0^2(Y, \mathcal{C}, \nu)$ . Recall that by  $G(g)$  we denote the cyclic space generated by  $g$ , *i.e.*

$$G(g) = \overline{\text{span}}\{g \circ S_h : h \in G\}.$$

The space  $L^2(X, \mathcal{B}, \mu) \otimes G(g)$  is  $T_{\varphi,S}$ -invariant. Indeed, we can naturally identify  $L^2(X \times Y, \mathcal{B} \otimes \mathcal{C}, \mu \otimes \nu)$  with the space  $L^2(X, \mathcal{B}, \mu; L^2(Y, \mathcal{C}, \nu))$  of square-integrable  $L^2(Y, \mathcal{C}, \nu)$ -valued functions on  $(X, \mathcal{B}, \mu)$ , and then  $L^2(X, \mathcal{B}, \mu) \otimes G(g)$  becomes the subspace of functions taking  $\mu$ -a.e. their values in  $G(g)$ . Now, if  $F(x, \cdot) \in G(g)$ , then  $(F \circ T_{\varphi,S})(x, \cdot) = F(Tx, \cdot) \circ S_{\varphi(x)} \in G(g)$ .

For each  $h \in G$ , as  $\sigma_{g \circ S_h, S} = \sigma_{g, S}$ , we have from (19)

$$\sigma_{f \otimes g, T_{\varphi,S}} = \sigma_{f \otimes (g \circ S_h), T_{\varphi,S}}.$$

Let  $\{f_n\}_{n \geq 0}$  be an orthonormal base in  $L^2(X, \mathcal{B}, \mu)$  with  $f_0 = 1$ . It is then clear that

$$\sigma_{T_{\varphi,S}|_{L^2(X, \mu) \otimes G(g)}} = \sum_{n \geq 0} 2^{-n} \sigma_{f_n \otimes g, T_{\varphi,S}}. \quad (20)$$

Therefore, by the proof of Lemma 3, we obtain the following.

**Lemma 4** *We have*

$$\sigma_{T_{\varphi,S}|_{L^2(X, \mu) \otimes G(g)}} = \int_{\hat{G}} \sigma_{V_{\chi \circ \varphi}} d\sigma_{g,S}(\chi).$$

□

### 3.3 Ergodicity of $T_{\varphi,S}$

We assume here that  $T$  is ergodic. Let us first notice that, then

$$\chi \in \Lambda_{\varphi} \text{ if and only if } \sigma_{V_{\chi \circ \varphi}}(\{1\}) > 0 \quad (21)$$

or, more generally, that

$$\begin{aligned} &\text{The cocycle } \chi \circ \varphi \text{ is cohomologous to } e^{2\pi i t} \\ &\text{if and only if } \sigma_{V_{\chi \circ \varphi}}(\{e^{2\pi i t}\}) > 0. \end{aligned} \quad (22)$$

Indeed,  $\sigma_{V_{\chi \circ \varphi}}(\{e^{2\pi it}\}) > 0$  if and only if  $e^{2\pi it}$  is an eigenvalue of  $V_{\chi \circ \varphi}$  and any eigenfunction corresponding to this eigenvalue will have constant modulus and so, up to normalization, be a transfer function  $j$  in the cohomology equation  $\chi \circ \varphi = e^{2\pi it} \cdot j/j \circ T$ .

The result below has already been proved in [21]. We give however a shorter proof.

**Proposition 5 ([21])**  *$T_{\varphi, \mathcal{S}}$  is ergodic if and only if  $T$  is ergodic and  $\sigma_{\mathcal{S}}(\Lambda_{\varphi}) = 0$ .*

**Proof.**

It is clearly necessary that  $T$  be ergodic. Then, by Lemma 3,  $\sigma_{T_{\varphi, \mathcal{S}}}(\{1\}) = 0$  if and only if  $\sigma_{V_{\chi \circ \varphi}}(\{1\}) = 0$  for  $\sigma_{\mathcal{S}}$ -a.e.  $\chi \in \widehat{G}$  and therefore, in view of (21), if and only if  $\sigma_{\mathcal{S}}(\Lambda_{\varphi}) = 0$ .  $\square$

**Remark 3** Let us notice that  $\sigma_{\mathcal{S}}(\Lambda_{\varphi}) = 0$  implies that  $\sigma_{\mathcal{S}}(\{1\}) = 0$ . Indeed a necessary condition for ergodicity of  $T_{\varphi, \mathcal{S}}$  is the ergodicity property of  $\mathcal{S}$  itself.

### 3.4 Eigenvalues of $T_{\varphi, \mathcal{S}}$

Assume now that  $T_{\varphi, \mathcal{S}}$  is ergodic. We will determine its eigenvalues (and eigenfunctions). Let us fix  $t \in [0, 1)$  and set

$$A_t = \{\chi \in \widehat{G} : \chi \circ \varphi \text{ is cohomologous to } e^{2\pi it}\}.$$

Notice that  $A_t \subset \Sigma_{\varphi}$  and that if  $\chi \in A_t$  and  $\chi_1 \in \Lambda_{\varphi}$  then  $\chi\chi_1 \in A_t$ . Moreover, if  $\chi_1, \chi_2 \in A_t$  belong to  $A_t$  then  $\chi_1\overline{\chi_2} \in \Lambda_{\varphi}$ . It follows that  $A_t$  is a coset of  $\Lambda_{\varphi}$ .

Suppose that  $e^{2\pi it}$  is an eigenvalue of  $T_{\varphi, \mathcal{S}}$ , i.e.  $\sigma_{T_{\varphi, \mathcal{S}}}(\{e^{2\pi it}\}) > 0$  and let  $F$  be a corresponding eigenfunction. We shall assume that  $F$  is not a function of  $x$  alone (otherwise  $F$  is an eigenfunction of  $T$  and the result below is trivial). Then there exists  $g \in L_0^2(Y, \mathcal{C}, \nu)$  such that  $F$  is not orthogonal to  $L^2(X, \mathcal{B}, \mu) \otimes G(g)$ . Since the spectral measure of  $F$  is the Dirac measure at  $e^{2\pi it}$ , it follows from Lemma 4 that

$$\int_{\widehat{G}} \sigma_{V_{\chi \circ \varphi}} d\sigma_{g, \mathcal{S}}(\chi)(\{e^{2\pi it}\}) > 0.$$

The latter occurs if and only if  $\sigma_{g, \mathcal{S}}(\{\chi \in \widehat{G} : \sigma_{V_{\chi \circ \varphi}}(\{e^{2\pi it}\}) > 0\}) > 0$ , that is, by (22), if and only if  $\sigma_{g, \mathcal{S}}(A_t) > 0$ . Now  $A_t$  is a coset  $\chi_0\Lambda_{\varphi}$  of  $\Lambda_{\varphi}$

and there exists then a non-zero  $g_1 \in G(g) \cap H_{\chi_0 \Lambda_\varphi}$ . According to Lemma 2 (and Theorem 1), since  $T_{\varphi, \mathcal{S}}$  is ergodic and thus  $\sigma_{\mathcal{S}}(\Lambda_\varphi) = 0$ , it follows that  $g_1$  is an eigenfunction corresponding to an eigenvalue  $\chi \in A_t$ .

Let now  $f$  be a measurable function of modulus 1 satisfying  $\chi \circ \varphi \cdot f \circ T = e^{2\pi i t} \cdot f$   $\mu$ -a.e. As  $g_1 \circ S_{\varphi(x)} = \chi(\varphi(x)) \cdot g_1$ , we get

$$(f \otimes g_1) \circ T_{\varphi, \mathcal{S}} = (\chi \circ \varphi \cdot f \circ T) \otimes g_1 = e^{2\pi i t} \cdot (f \otimes g_1).$$

So,  $f \otimes g_1$  is an eigenfunction of  $T_{\varphi, \mathcal{S}}$  corresponding to  $e^{2\pi i t}$  and, since  $T_{\varphi, \mathcal{S}}$  is ergodic,  $F$  can be different from  $f \otimes g_1$  only by a multiplicative constant. Therefore we have proved the following.

**Proposition 6** *Assume that  $T_{\varphi, \mathcal{S}}$  is ergodic. Then the eigenfunctions of  $T_{\varphi, \mathcal{S}}$  are the functions of the form  $f \otimes g$ , where  $\chi \circ \varphi = e^{2\pi i t} \cdot f/f \circ T$ ,  $\chi$  is an eigenvalue of  $\mathcal{S}$  and  $g$  is an eigenfunction corresponding to  $\chi$ . In particular,  $e^{2\pi i t}$  ( $t \in [0, 1)$ ) is an eigenvalue of  $T_{\varphi, \mathcal{S}}$  if and only if there exists an eigenvalue of  $\mathcal{S}$  in  $A_t$ .  $\square$*

### 3.5 Weak mixing and relative weak mixing

A characterization of the weak mixing property for  $T_{\varphi, \mathcal{S}}$  is a direct corollary of Proposition 6.

**Corollary 3**  *$T_{\varphi, \mathcal{S}}$  is weakly mixing if and only if it is ergodic,  $T$  is weakly mixing and  $\mathcal{S}$  has no eigenvalues in  $\Sigma_\varphi$ .  $\square$*

**Remark 4** Notice that this corollary generalizes the well-known criterion for weak mixing property of Abelian compact group extensions.

Let us pass to a characterization of the relative weak mixing property. We still assume that  $T_{\varphi, \mathcal{S}}$  is ergodic.

Let us first notice that the relative product of  $T_{\varphi, \mathcal{S}}$  with itself over the factor  $T$  is isomorphic to  $T_{\varphi, \mathcal{S} \times \mathcal{S}}$ , where  $\mathcal{S} \times \mathcal{S}$  stands for the diagonal action  $g \mapsto S_g \times S_g$  of  $G$  on  $(Y \times Y, \mathcal{C} \otimes \mathcal{C}, \nu \otimes \nu)$ . So  $T_{\varphi, \mathcal{S}}$  is relatively weakly mixing over  $T$  if and only if  $T_{\varphi, \mathcal{S} \times \mathcal{S}}$  is ergodic.

Since  $\sigma_{\mathcal{S} \times \mathcal{S}} = \sigma_{\mathcal{S}} + \sigma_{\mathcal{S}} * \sigma_{\mathcal{S}}$ , it follows from Proposition 5 that  $T_{\varphi, \mathcal{S}}$  is relatively weakly mixing over  $T$  if and only if  $\sigma_{\mathcal{S}}(\Lambda_\varphi) + \sigma_{\mathcal{S}} * \sigma_{\mathcal{S}}(\Lambda_\varphi) = 0$ . The latter statement is equivalent to saying that  $\sigma_{\mathcal{S}}(\chi \Lambda_\varphi) = 0$  for each  $\chi \in \widehat{G}$ . This has already been proved in [21] but now we have Lemma 2 at our disposal which finally improves and clarifies the result: Since  $T_{\varphi, \mathcal{S}}$  is

ergodic, we have  $\sigma_{\mathcal{S}}(\Lambda_{\varphi}) = 0$ , and  $\sigma_{\mathcal{S}}(\chi_0 \Lambda_{\varphi}) > 0$  for some  $\chi_0$  if and only if  $\mathcal{S}$  has an eigenvalue.

**Proposition 7**  *$T_{\varphi, \mathcal{S}}$  is relatively weakly mixing over  $T$  if and only if it is ergodic and  $\mathcal{S}$  is weakly mixing.*  $\square$

### 3.6 Relative Kronecker factor of $T_{\varphi, \mathcal{S}}$ over $T$

Denote by  $\mathcal{K}(\mathcal{S}) \subset \mathcal{C}$  the Kronecker factor of  $\mathcal{S}$ , i.e. the factor generated by the eigenfunctions of the unitary action of  $\mathcal{S}$ . If  $g, h$  are eigenfunctions of  $\mathcal{S}$  (corresponding to  $\chi$  and  $\chi'$  respectively) then

$$(g \otimes h) \circ T_{\varphi, \mathcal{S} \times \mathcal{S}}(x, y, y') = \chi(\varphi(x))\chi'(\varphi(x)) \cdot (g \otimes h)(y, y').$$

It follows that  $\mathcal{B} \otimes \mathcal{K}(\mathcal{S})$  is contained in the relative Kronecker factor of  $T_{\varphi, \mathcal{S}}$  over  $T$  (cf. (3) and (4) for  $n = 1$ ). In fact, we have the following.

**Proposition 8** *Assume that  $T_{\varphi, \mathcal{S}}$  is ergodic. The relative Kronecker factor of  $T_{\varphi, \mathcal{S}}$  over  $T$  is equal to  $\mathcal{B} \otimes \mathcal{K}(\mathcal{S})$ .*

**Proof.**

Assume that  $F \in L^2(X \times Y \times Y, \mu \otimes \nu \otimes \nu)$  is a  $T_{\varphi, \mathcal{S} \times \mathcal{S}}$ -invariant function. Take  $g, h \in L^2(Y, \mathcal{C}, \nu)$  and suppose that  $F$  is not orthogonal to  $L^2(X, \mathcal{B}, \mu) \otimes G(g \otimes h)$ . Then, proceeding as in the proof of Proposition 6, we obtain that  $\sigma_{g \otimes h, \mathcal{S} \times \mathcal{S}}(\Lambda_{\varphi}) = \sigma_{g, \mathcal{S}} * \sigma_{h, \mathcal{S}}(\Lambda_{\varphi}) > 0$ . Therefore  $\sigma_{g, \mathcal{S}}(\chi \Lambda_{\varphi}) > 0$  for some  $\chi \in \widehat{G} \setminus \{1\}$  (and the same holds for  $h$ ). By Lemma 2, remembering that  $T_{\varphi, \mathcal{S}}$  is ergodic,  $g$  is not orthogonal to an eigenfunction of  $\mathcal{S}$  from  $H_{\chi \Lambda}$ . It follows that if  $\{g_i\}_{i \geq 0}$  stands for an orthonormal base of  $L^2(\mathcal{K}(\mathcal{S}))$ , where each  $g_i$  is an eigenfunction corresponding to  $\chi_i$ ,  $i \geq 0$ , then

$$F(x, y, y') = \sum_{i, j \geq 0} a_{ij}(x) g_i(y) g_j(y'),$$

where  $\chi_i \cdot \chi_j \in \Lambda_{\varphi}$ , whenever  $a_{ij} \neq 0$  (in fact  $\chi_j = \overline{\chi_i}$  since there is at most one eigenvalue in a coset  $\chi \Lambda_{\varphi}$ ). Fix any function  $J = J(x, y') \in L^2(X \times Y, \mu \otimes \nu)$ . Then, for each  $i, j \geq 0$  the function given by

$$((a_{ij} \otimes g_i \otimes g_j) * J)(x, y) := \int_Y a_{ij}(x) g_i(y) g_j(y') J(x, y') d\nu(y')$$

is of the form  $A(x)g_i(y)$ , so it is measurable with respect to  $\mathcal{B} \otimes \mathcal{K}(\mathcal{S})$ . The result follows then directly from the description of the relative Kronecker factor given in [8], Theorem 6.13.  $\square$

**Remark 5** Proposition 8 yields another proof of the result about eigenfunctions of  $T_{\varphi, \mathcal{S}}$  when  $T_{\varphi, \mathcal{S}}$  is ergodic. Indeed, eigenfunctions are measurable with respect to the relative Kronecker factor. If, as before,  $\{g_i\}$  stands for an orthonormal base of eigenfunctions in  $L^2(\mathcal{K}(\mathcal{S}))$  and  $F \circ T_{\varphi, \mathcal{S}} = c \cdot F$ , then

$$F(x, y) = \sum_{i \geq 0} a_i(x) g_i(y), \quad F \circ T_{\varphi, \mathcal{S}}(x, y) = \sum_{i \geq 0} a_i(Tx) \chi_i(\varphi(x)) g_i(y),$$

so  $c \cdot a_i(x) = a_i(Tx) \cdot \chi_i(\varphi(x))$  for each  $i \geq 0$ .

### 3.7 Regularity of Rokhlin cocycles

When the group  $\Lambda_\varphi$  of a cocycle  $\varphi : X \rightarrow G$  is not trivial then, obviously, the skew product  $T_\varphi : (X \times G, \mu \otimes \lambda_G) \rightarrow (X \times G, \mu \otimes \lambda_G)$ ,  $T_\varphi(x, g) = (Tx, \varphi(x) \cdot g)$ , is not ergodic, but  $\varphi$  need not be a coboundary. We will show however in this section that on the level of Rokhlin cocycles, that is when considering the cocycle  $x \mapsto S_{\varphi(x)}$ , it must be a coboundary as soon as  $\sigma_{\mathcal{S}}$  is concentrated on  $\Lambda_\varphi$ <sup>20</sup>. In the general case, we denote by  $\mathcal{A}_{\Lambda_\varphi}$  be the  $\mathcal{S}$ -factor corresponding to  $\Lambda_\varphi$  according to Corollary 1, *i.e.*  $L^2(\mathcal{A}_{\Lambda_\varphi}) = H_{\Lambda_\varphi}$ , and we will show that  $x \mapsto S_{\varphi(x)}|_{\mathcal{A}_{\Lambda_\varphi}}$  is a coboundary.

We show firstly that, when  $T$  is ergodic,  $\mathcal{B} \otimes \mathcal{A}_{\Lambda_\varphi}$  contains the factor of  $T_{\varphi, \mathcal{S}}$ -invariant sets.

**Lemma 5** *Assume that  $T$  is ergodic. Every  $T_{\varphi, \mathcal{S}}$ -invariant function  $F$  in  $L^2(X \times Y, \mathcal{B} \otimes \mathcal{C}, \mu \otimes \nu)$  is  $\mathcal{B} \otimes \mathcal{A}_{\Lambda_\varphi}$ -measurable.*

**Proof.**

Given any  $g \in L^2_0(Y, \mathcal{C}, \nu)$ , the projection of  $F$  on the  $T_{\varphi, \mathcal{S}}$ -invariant subspace  $L^2(X, \mathcal{B}, \mu) \otimes G(g)$  is still  $T_{\varphi, \mathcal{S}}$ -invariant. If this projection is non-zero, the maximal spectral type of  $T_{\varphi, \mathcal{S}}$  on  $L^2(X, \mathcal{B}, \mu) \otimes G(g)$  must have an atom at 1. Then, by Lemma 4,  $\sigma_{g, \mathcal{S}}(\{\chi \in \widehat{G} : \sigma_{V_{\chi \circ \varphi}}(\{1\}) > 0\}) > 0$  and so, since  $T$  is ergodic,  $\sigma_{g, \mathcal{S}}(\Lambda_\varphi) > 0$  in view of (21). Since  $g$  was arbitrary, it follows  $F \in L^2(X, \mathcal{B}, \mu) \otimes H_{\Lambda_\varphi} = L^2(X \times Y, \mathcal{B} \otimes \mathcal{A}_{\Lambda_\varphi}, \mu \otimes \nu)$ .  $\square$

We give now a short description of the action of  $T_{\varphi, \mathcal{S}}$  on  $L^2(X, \mathcal{B}, \mu) \otimes G(g)$  for a non-zero  $g \in L^2(X, \mathcal{B}, \mu)$ , which will shed some light on the proof of Proposition 9 below. We identify naturally  $L^2(X, \mathcal{B}, \mu) \otimes G(g)$  to  $L^2(X, \mathcal{B}, \mu; G(g))$ . We may furthermore replace  $G(g)$  by  $L^2(\widehat{G}, \mathcal{B}(\widehat{G}), \sigma_{g, \mathcal{S}})$

---

<sup>20</sup>When  $\Lambda_\varphi$  is uncountable then there is always a weakly mixing Gaussian action  $\mathcal{S}$  such that  $\sigma_{\mathcal{S}}$  is concentrated on  $\Lambda_\varphi$ .

through the canonical spectral isomorphism, which we denote by  $I$ . We shall determine the unitary operator  $V_\varphi$  on  $L^2(X, \mathcal{B}, \mu; L^2(\widehat{G}, \mathcal{B}(\widehat{G}), \sigma_{g,S})) = L^2(X \times \widehat{G}, \mathcal{B} \otimes \mathcal{B}(\widehat{G}), \mu \otimes \sigma_{g,S})$  which corresponds to  $T_{\varphi,S}$  acting on  $L^2(X, \mathcal{B}, \mu) \otimes G(g)$ .

Let  $\tilde{F} \in L^2(X \times \widehat{G}, \mathcal{B} \otimes \mathcal{B}(\widehat{G}), \mu \otimes \sigma_{g,S})$  correspond to  $F \in L^2(X, \mathcal{B}, \mu) \otimes G(g)$  - i.e.  $\tilde{F}(x, \cdot) = I(F(x, \cdot))$  for  $\mu$ -a.e.  $x$ . We have, for  $\mu$ -a.e.  $x$ ,

$$(V_\varphi \tilde{F})(x, \cdot) = I(F(Tx, S_{\varphi(x)}(\cdot))).$$

Now, if  $h \in G$ , the image of  $g \circ S_h$  under  $I$  is  $\tilde{h}$ , where  $\tilde{h}(\chi) = \chi(h)$  for  $\chi \in \widehat{G}$ . If we take  $F$  of the form  $F = f \otimes g \circ S_h$  then  $\tilde{F} = f \otimes \tilde{h}$  and

$$V_\varphi(f \otimes \tilde{h})(x, \cdot) = f(Tx) \cdot I(g \circ S_{\varphi(x) \cdot h}) = \chi(\varphi(x)) \cdot (f \otimes \tilde{h})(Tx, \cdot).$$

It follows that for each  $\tilde{F} \in L^2(X \times \widehat{G}, \mu \otimes \sigma_{g,S})$ ,  $\mu \otimes \sigma_{g,S}$ -a.e.  $(x, \chi) \in X \times \widehat{G}$ ,

$$(V_\varphi \tilde{F})(x, \chi) = \chi(\varphi(x)) \tilde{F}(Tx, \chi). \quad (23)$$

We come to the result announced at the beginning of this section.

**Proposition 9** *The Rokhlin cocycle  $x \mapsto S_{\varphi(x)}|_{\mathcal{A}_{\Lambda_\varphi}}$  is a coboundary. In other words  $T_{\varphi,S}|_{\mathcal{B} \otimes \mathcal{A}_{\Lambda_\varphi}}$  is relatively isomorphic to  $T \times id_Y|_{\mathcal{B} \otimes \mathcal{A}_{\Lambda_\varphi}}$ .*

**Proof.**

Let  $\sigma_S|_{\Lambda_\varphi}$  be the spectral type of  $\mathcal{S}|_{L^2(\mathcal{A}_{\Lambda_\varphi})}$  (i.e.  $\sigma_S|_{\Lambda_\varphi}$  is  $\sigma_S$  restricted to  $\Lambda_\varphi$ ). When  $\chi \in \Lambda_\varphi$ , the cocycle  $\chi \circ \varphi$  is a coboundary, that is  $\chi \circ \varphi = f/f \circ T$   $\mu$ -a.e. for some measurable  $f$  of modulus 1. In fact there exists a measurable selector of transfer functions defined  $\sigma_S|_{\Lambda_\varphi}$ -a.e. (see e.g. [17]). This means that there exists a measurable function  $\tilde{F}$  of modulus 1 on  $X \times \Lambda_\varphi$  such that

$$\chi(\varphi(x)) = \tilde{F}(x, \chi) / \tilde{F}(Tx, \chi) \text{ for } \mu \otimes \sigma_S|_{\Lambda_\varphi}\text{-a.a. } (x, \chi) \quad (24)$$

(so, for every  $g \in L^2(\mathcal{A}_{\Lambda_\varphi})$ , the function  $F \in L^2(X, \mathcal{B}, \mu) \otimes G(g)$  corresponding to  $\tilde{F}$  is  $T_{\varphi,S}$ -invariant).

Given  $x \in X$ , for every  $g \in L^2(\mathcal{A}_{\Lambda_\varphi})$  the action of  $S_{\varphi(x)}$  on  $G(g)$  corresponds through the spectral isomorphism to the multiplication by the function  $\chi \mapsto \chi(\varphi(x))$ . On the other hand, by the canonical action of  $L^\infty(\widehat{G}, \mathcal{B}(\widehat{G}), \sigma_S|_{\Lambda_\varphi})$  on  $L^2(\mathcal{A}_{\Lambda_\varphi})$ , there also exists a unitary operator  $W_x$  on  $L^2(\mathcal{A}_{\Lambda_\varphi})$  whose restriction to each  $G(g)$ , for  $g \in L^2(\mathcal{A}_{\Lambda_\varphi})$ , corresponds to the

multiplication by the unit-modulus function  $\tilde{F}(x, \cdot)$ . Then the equality (24) yields

$$S_{\varphi(x)}|_{L^2(\mathcal{A}_{\Lambda_\varphi})} = W_x W_{Tx}^{-1} \text{ for } \mu\text{-a.a. } x.$$

So, the cocycle  $x \mapsto S_{\varphi(x)}|_{L^2(\mathcal{A}_{\Lambda_\varphi})}$  is a  $T$ -coboundary as a cocycle taking values in  $\mathcal{U}(L^2(\mathcal{A}_{\Lambda_\varphi}))$ .

Finally,  $\text{Aut}(\mathcal{A}_{\Lambda_\varphi})$  is naturally identified to a closed subgroup of the Polish group  $\mathcal{U}(L^2(\mathcal{A}_{\Lambda_\varphi}))$ . Since  $x \mapsto S_{\varphi(x)}|_{L^2(\mathcal{A}_{\Lambda_\varphi})}$  takes its values in  $\text{Aut}(\mathcal{A}_{\Lambda_\varphi})$ , it is still a coboundary as an  $\text{Aut}(\mathcal{A}_{\Lambda_\varphi})$ -valued cocycle.  $\square$

In view of Proposition 5 we obtain the following result.

**Corollary 4** *If  $T$  is ergodic and  $T_{\varphi, \mathcal{S}}$  is not ergodic then there is a non-trivial factor  $\mathcal{A}$  of  $\mathcal{S}$  such that  $T_{\varphi, \mathcal{S}}|_{\mathcal{B} \otimes \mathcal{A}}$  is relatively isomorphic to  $T \times \text{id}_Y|_{\mathcal{B} \otimes \mathcal{A}}$ .*  $\square$

### 3.8 Lifting mild mixing property

In this section we will show that the triviality of the Rokhlin cocycle described in Proposition 9 also takes place when dealing with the mild mixing property, and we give necessary and sufficient conditions in order that the mild mixing property lift from  $T$  to  $T_{\varphi, \mathcal{S}}$ . Recall that  $T$  is mildly mixing if  $T$  has no non-trivial rigid factors and that a factor  $\mathcal{A}$  of  $T$  is rigid if and only if the spectral type of  $T|_{\mathcal{A}}$  is a Dirichlet measure.

We will need the following.

**Lemma 6** *Assume that  $T$  is mildly mixing. If  $\xi : X \rightarrow \mathbb{T}$  is a cocycle and  $T_\xi$  has a non-trivial rigid factor  $\mathcal{A} \subset \mathcal{B} \otimes \mathcal{B}(\mathbb{T})$  then there exist a factor  $\mathcal{A}'$  of  $T_\xi$  containing  $\mathcal{A}$  and a mixing sequence  $(q_n)$  for  $T$  such that  $(T_\xi)^{q_n} \rightarrow E(\cdot|_{\mathcal{A}'})$ .*

**Proof.**

From Proposition 2 there exist a factor  $\mathcal{A}'$  of  $T_\xi$  containing  $\mathcal{A}$  and a rigid sequence  $(q_n)$  for  $T_\xi|_{\mathcal{A}'}$  such that

$$(T_\xi)^{q_n} \rightarrow E(\cdot|_{\mathcal{A}'}). \quad (25)$$

It remains to show that  $(q_n)$  is a mixing sequence for  $T$ . But, since we assume that  $T$  is mildly mixing, no spectral measure of a function in  $L_0^2(X, \mathcal{B}, \mu)$  is a Dirichlet measure, while the maximal spectral type of  $T_\xi$  on  $L^2(\mathcal{A}')$  is a Dirichlet measure. Thus  $L_0^2(\mathcal{B} \otimes \{\emptyset, \mathbb{T}\}) \perp L^2(\mathcal{A}')$  and the result follows from (25).  $\square$



**Proposition 10** *Assume that  $T$  is mildly mixing. If  $\sigma_S(\Sigma_\varphi) = 0$ , then  $T_{\varphi,S}$  is also mildly mixing.*

**Proof.**

First, we claim that, if some positive measure  $\sigma_1 \ll \sigma_{V_{\chi \circ \varphi}}$  is a Dirichlet measure, then  $\chi \in \Sigma_\varphi$ .

Indeed, then there exists  $f \in L^2(X, \mathcal{B}, \mu)$  such that  $\sigma_{f, V_{\chi \circ \varphi}} = \sigma_1$ . If we consider  $F = f \otimes z \in L^2(X \times \mathbb{T}, \mu \otimes \lambda)$  (here  $z$  denotes the identity function from  $\mathbb{T}$  to  $\mathbb{C}$ ), we have  $F \circ T_{\chi \circ \varphi}^n = V_{\chi \circ \varphi}^n f \otimes z$  for all  $n \in \mathbb{Z}$  and it follows that  $\sigma_{F, T_{\chi \circ \varphi}} = \sigma_1$ . Since  $\sigma_1$  is a Dirichlet measure,  $F$  is measurable with respect to some rigid factor  $\mathcal{A}$  of  $T_{\chi \circ \varphi}$ .

In view of Lemma 6, there exist a factor  $\mathcal{A}'$  of  $T_{\chi \circ \varphi}$  containing  $\mathcal{A}$  and a mixing sequence  $(q_n)$  for  $T$  such that  $T_{\chi \circ \varphi}^{q_n} \rightarrow E(\cdot | \mathcal{A}')$ . In particular  $F \circ T_{\chi \circ \varphi}^{q_n} \rightarrow F$  and  $V_{\chi \circ \varphi}^{q_n} f \rightarrow f$ , so  $(q_n)$  is not a mixing sequence for  $V_{\chi \circ \varphi}$ . But Corollary 2 implies then that the cocycle  $\chi \circ \varphi$  is a quasi-coboundary, in other words  $\chi \in \Sigma_\varphi$ , which proves our claim.

It remains to show that there is no positive measure  $\sigma \ll \sigma_{T_{\varphi,S}}$  which is a Dirichlet measure. Suppose the contrary: then, for some sequence  $n_j \rightarrow \infty$ ,  $z^{n_j} \rightarrow 1$   $\sigma$ -a.e. It follows that if we set  $A = \{z \in \mathbb{T} : z^{n_j} \rightarrow 1\}$ , then  $\sigma_{T_{\varphi,S}}(A) > 0$ . Since  $T$  is mildly mixing and thus  $\sigma_T(A) = 0$ , it follows by Lemma 3 that  $\sigma_S(\{\chi \in \widehat{G} : \sigma_{V_{\chi \circ \varphi}}(A) > 0\}) > 0$ .

But clearly if  $\sigma_{V_{\chi \circ \varphi}}(A) > 0$ , then the positive measure  $\sigma_{V_{\chi \circ \varphi}}|_A$  is Dirichlet, hence  $\chi$  must belong to  $\Sigma_\varphi$  by the first part of the proof. Since  $\sigma_S(\Sigma_\varphi) = 0$ , we obtain a contradiction.  $\square$

**Remark 6** Supposing only that  $T$  is mildly mixing, we get that each rigid function of  $T_{\varphi,S}$  belongs to  $L^2(X, \mathcal{B}, \mu) \otimes H_{\Sigma_\varphi}$ . Indeed, if  $\sigma_{g,S}(\Sigma_\varphi) = 0$ , in view of Lemma 4, the same proof gives that there is no Dirichlet measure  $\sigma \ll \sigma_{T_{\varphi,S}}|_{L^2(X, \mathcal{B}, \mu) \otimes G(g)}$ .

Let us denote by  $\mathcal{A}_{\Sigma_\varphi}$  is the factor of  $\mathcal{S}$  corresponding to the saturated group  $\Sigma_\varphi$  according to Corollary 1, so that  $H_{\Sigma_\varphi} = L^2(\mathcal{A}_{\Sigma_\varphi})$ . In other words,

$$\text{each rigid factor of } T_{\varphi,S} \text{ is contained in } \mathcal{B} \otimes \mathcal{A}_{\Sigma_\varphi}. \quad (26)$$

**Proposition 11** *Assume that  $T$  is weakly mixing (and  $\sigma_S(\Sigma_\varphi) > 0$ ). Then there exists an automorphism  $U$  of  $(Y|_{\mathcal{A}_{\Sigma_\varphi}}, \mathcal{A}_{\Sigma_\varphi}, \nu|_{\mathcal{A}_{\Sigma_\varphi}})$  such that  $T_{\varphi,S}|_{\mathcal{B} \otimes \mathcal{A}_{\Sigma_\varphi}}$  is isomorphic to  $T \times U$ .*

**Proof.**

By definition, if  $\chi \in \Sigma_\varphi$  the cocycle  $\chi \circ \varphi$  is cohomologous to a constant  $e^{2\pi i t}$ . However  $T$  is weakly mixing, so this constant is unique, hence we can

write  $t = t(\chi)$  ( $t(\chi) \in [0, 1)$ ). Then, as in the case of  $\Lambda_\varphi$ , there exists a measurable selector of transfer functions  $\chi \mapsto \tilde{F}(\cdot, \chi)$  defined on  $\sigma_S|_{\Sigma_\varphi}$  a.e., equivalently a measurable function  $\tilde{F}$  of modulus 1 on  $X \times \Sigma_\varphi$  such that

$$\chi(\varphi(x)) = e^{2\pi i t(\chi)} \tilde{F}(x, \chi) / \tilde{F}(Tx, \chi) \text{ for } \mu \otimes \sigma_S|_{\Sigma_\varphi}\text{-a.a. } (x, \chi). \quad (27)$$

In particular,  $e^{2\pi i t(\chi)} = u(\chi)$ ,  $\sigma_S|_{\Sigma_\varphi}$ -a.e., where  $u$  is a measurable function of modulus 1 on  $\Sigma_\varphi$ .

Then, as in the proof of Proposition 9, we deduce that there exist unitary operators  $U$  and  $W_x$  ( $x \in X$ ) of  $L^2(\mathcal{A}_{\Sigma_\varphi})$  corresponding to the multiplication by  $u$  and  $\tilde{F}(x, \cdot)$  respectively, so that

$$S_{\varphi(x)}|_{\Sigma_\varphi} = UW_x W_{Tx}^{-1} \quad \mu\text{-a.e. } x. \quad (28)$$

We have to show that  $U$  corresponds to an automorphism. Let us consider the  $(T \times T, \mu \otimes \mu)$ -cocycle  $(x_1, x_2) \mapsto S_{\varphi(x_2)} S_{\varphi(x_1)}^{-1}|_{\Sigma_\varphi}$ . In view of (28) (and the fact that the operators under consideration commute), it is a coboundary with the transfer operator map  $(x_1, x_2) \mapsto W_{x_2} W_{x_1}^{-1}$ , whence, as in the proof of Proposition 9, it is also a coboundary as a cocycle with values in  $\text{Aut}(\mathcal{A}_{\Sigma_\varphi})$ . Thus there exists a measurable map  $(x_1, x_2) \mapsto V_{x_1, x_2} \in \text{Aut}(\mathcal{A}_{\Sigma_\varphi})$  with

$$S_{\varphi(x_2)} S_{\varphi(x_1)}^{-1}|_{\Sigma_\varphi} = V_{x_1, x_2} V_{Tx_1, Tx_2}^{-1} \quad \text{for } \mu \otimes \mu\text{-a.a. } (x_1, x_2).$$

Since  $T$  is weakly mixing,  $T \times T$  is ergodic and therefore the two transfer operator maps must coincide up to a constant. More precisely,  $W_{x_1} W_{x_2}^{-1} V_{x_1, x_2}$  is  $T \times T$ -invariant, so there exists a unitary operator  $V$  of  $L^2(\mathcal{A}_{\Sigma_\varphi})$  such that

$$W_{x_1} W_{x_2}^{-1} V_{x_1, x_2} = V \quad \text{for } \mu \otimes \mu\text{-a.a. } (x_1, x_2).$$

By selecting  $x_1$  so that the above equality is true for  $\mu$ -a.e.  $x_2$ , we obtain

$$V_{x_1, x} = W_x W_{x_1}^{-1} V \quad \text{for } \mu\text{-a.a. } x.$$

Then the map  $x \mapsto V_{x_1, x} \in \text{Aut}(\mathcal{A}_{\Sigma_\varphi})$  is also a transfer operator map for the equation (28):

$$UV_{x_1, x} V_{x_1, Tx}^{-1} = UW_x W_{Tx}^{-1} = S_{\varphi(x)}|_{\Sigma_\varphi} \quad \mu\text{-a.e. } x.$$

Therefore  $U \in \text{Aut}(\mathcal{A}_{\Sigma_\varphi})$ ,  $x \mapsto S_{\varphi(x)}$  is cohomologous to the constant  $U$  in  $\text{Aut}(\mathcal{A}_{\Sigma_\varphi})$ , and the result follows.  $\square$

**Corollary 5** *Assume that  $T$  is mildly mixing. Then  $T_{\varphi, \mathcal{S}}$  is not mildly mixing if and only if there exists a non-trivial factor  $\mathcal{A}$  of  $\mathcal{S}$  and an automorphism  $U$  of  $(Y|_{\mathcal{A}}, \mathcal{A}, \nu|_{\mathcal{A}})$  which is not mildly mixing such that  $T_{\varphi, \mathcal{S}}|_{\mathcal{B} \otimes \mathcal{A}}$  is isomorphic to  $T \otimes U$ .*

**Proof.**

In view of (26), if  $T_{\varphi, \mathcal{S}}$  is not mildly mixing, neither is  $T_{\varphi, \mathcal{S}}|_{\mathcal{B} \otimes \mathcal{A}_{\Sigma_\varphi}}$  and we apply Proposition 11. Then  $T \times U$  is not mildly mixing and, as  $T$  is mildly mixing,  $U$  cannot be mildly mixing. The other direction is clear.  $\square$

**Remark 7** It turns out that in Corollary 5 we can replace  $U$  non mildly mixing by  $U$  rigid. Indeed, in the proof of Proposition 11,  $U$  corresponds to the multiplication by  $u(\chi)$ , and given a rigid factor  $\mathcal{A}$  of  $U$  defined by  $U^{n_j} h \rightarrow h$  for some sequence  $(n_j)$ , we have that  $L^2(\mathcal{A})$  is the spectral subspace of  $\mathcal{S}|_{\mathcal{A}_{\Sigma_\varphi}}$  corresponding to  $\{\chi \in \Sigma_\varphi : u(\chi)^{n'_j} \rightarrow 1\}$  for some subsequence; in particular,  $\mathcal{A}$  is also  $\mathcal{S}$ -invariant. Moreover, the cocycle  $S_{\varphi(x_2)} S_{\varphi(x_1)}^{-1}|_{\Sigma_\varphi}$  is still cohomologous to the constant  $U$  in the closed subgroup of  $\text{Aut}(\mathcal{A}_{\Sigma_\varphi})$  of all automorphisms corresponding to multiplications by unit-modulus functions (in the spectral representation of  $\mathcal{S}|_{\mathcal{A}_{\Sigma_\varphi}}$ ). So, the automorphisms  $V_{x_1, x_2}$  can be taken in this subgroup and hence preserving the invariant subspaces of  $\mathcal{S}$ . Then  $\mathcal{B} \otimes \mathcal{A}$  is preserved by the conjugation automorphism and we have relative isomorphism of  $T_{\varphi, \mathcal{S}}|_{\mathcal{B} \otimes \mathcal{A}}$  with  $T \otimes U|_{\mathcal{B} \otimes \mathcal{A}}$ .

We now show that under the recurrence property of  $\varphi$  the converse of Proposition 10 holds.

Let  $\sigma \in M^+(\widehat{G})$ . Denote by  $\mathcal{U}(\sigma)$  the group of measurable functions of modulus 1 defined on  $\widehat{G}$ , modulo equality  $\sigma$ -a.e. We endow  $\mathcal{U}(\sigma)$  with the  $L^2(\sigma)$ -topology, which makes it a Polish group. Given  $g \in G$ , we still denote by  $\tilde{g}$  the function  $\chi \mapsto \chi(g)$  taken as an element of  $\mathcal{U}(\sigma)$ . Then we define  $\varphi_\sigma$  from  $X$  to  $\mathcal{U}(\sigma)$  by setting

$$\varphi_\sigma(x)(\chi) = \chi(\varphi(x)) \text{ for each } \chi \in \widehat{G},$$

*i.e.*  $\varphi_\sigma$  is the composition of  $\varphi$  and of the map  $g \mapsto \tilde{g}$ . As the latter map is a continuous group homomorphism, it is clear from the definition (15) that

$$\text{if } \varphi \text{ is recurrent then so is } \varphi_\sigma. \quad (29)$$

**Proposition 12** *If  $T$  is mildly mixing, then  $T_{\varphi, \mathcal{S}}$  is mildly mixing if and only if  $\sigma(\Sigma_\varphi) = 0$  for each positive measure  $\sigma \ll \sigma_{\mathcal{S}}$  such that the cocycle  $\varphi_\sigma$  is recurrent.*

*In particular, if  $\varphi$  is recurrent and  $T_{\varphi, \mathcal{S}}$  is mildly mixing then  $\sigma_{\mathcal{S}}(\Sigma_\varphi) = 0$ .*

**Proof.**

We keep the notation as in the proof of Proposition 11:  $U$  is the automorphism of  $(Y|_{\mathcal{A}_{\Sigma_\varphi}}, \mathcal{A}_{\Sigma_\varphi}, \nu|_{\mathcal{A}_{\Sigma_\varphi}})$  corresponding to the unit-modulus function  $u$  on  $\Sigma_\varphi$ , and the equation (27) may now be written as

$$\varphi_\sigma(\chi) = u(\chi)\tilde{F}(x, \chi)/\tilde{F}(Tx, \chi) \text{ for } \mu \otimes \sigma_{\mathcal{S}}|_{\Sigma_\varphi}\text{-a.a. } (x, \chi).$$

Suppose that  $\varphi_\sigma$  is recurrent for some positive measure  $\sigma \ll \sigma_{\mathcal{S}}$  with  $\sigma(\Sigma_\varphi) > 0$ . We can then assume that  $0 < \sigma \ll \sigma_{\mathcal{S}}|_{\Sigma_\varphi}$ . Then the constant cocycle  $u$  restricted to  $\mathcal{U}(\sigma)$  is cohomologous to  $\varphi_\sigma$  and it is also recurrent by (16). Thus, in view of (17), there is a sequence  $(n_j)$  with  $u^{n_j} \rightarrow 1$  in  $\mathcal{U}(\sigma)$ , whence  $U^{n_j}h \rightarrow h$  for each function  $h$  such that  $\sigma_{h, \mathcal{S}} \ll \sigma$ . It follows that  $U$  is not mildly mixing.

For the other direction, if  $T_{\varphi, \mathcal{S}}$  is not mildly mixing, then  $U$  is not mildly mixing and we find conversely that  $u$  restricted to  $\mathcal{U}(\sigma_{h, \mathcal{S}})$  is recurrent, for some non-zero  $h \in L_0^2(\mathcal{A}_{\Sigma_\varphi})$ . Then  $\varphi_{\sigma_{h, \mathcal{S}}}$  is also recurrent and  $\sigma_{h, \mathcal{S}}(\Sigma_\varphi) > 0$ .

The second assertion follows then from (29): if  $\varphi$  is recurrent, then the cocycle  $\varphi_{\sigma_{\mathcal{S}}}$  is also recurrent.  $\square$

### 3.9 Lifting mixing and multiple mixing

We give here two corollaries of results from [22].

**Proposition 13 ([22])** *Assume that  $T$  is mixing. If  $\sigma_{\mathcal{S}}(\Sigma_\varphi) = 0$  then  $T_{\varphi, \mathcal{S}}$  is mixing. Conversely, if  $\sigma_{\mathcal{S}}(\Sigma_\varphi) > 0$  and  $\varphi$  is recurrent then  $T_{\varphi, \mathcal{S}}$  is not mixing.*  $\square$

**Corollary 6** *Assume that  $T$  is mixing. Then  $T_{\varphi, \mathcal{S}}$  is not mixing if and only if there exists a non-trivial factor  $\mathcal{A}$  of  $\mathcal{S}$  and an automorphism  $U$  of  $(Y|_{\mathcal{A}}, \mathcal{A}, \nu|_{\mathcal{A}})$  which is not mixing such that  $T_{\varphi, \mathcal{S}}|_{\mathcal{B} \otimes \mathcal{A}}$  is isomorphic to  $T \otimes U$ .*

**Proof.**

The proof of Proposition 13 in [22] (Theorem 7.1) shows actually that if  $T$  is mixing and  $\sigma_{h, \mathcal{S}}(\Sigma_\varphi) = 0$ , then  $\hat{\sigma}_{f \otimes h, T_{\varphi, \mathcal{S}}}(n) \rightarrow 0$  for each  $f \in L^2(X, \mathcal{B}, \mu)$ .

Therefore if  $T_{\varphi, \mathcal{S}}$  is not mixing, we must have a function  $F \in L_0^2(X, \mathcal{B}, \mu) \otimes H_{\Sigma_\varphi}$  whose spectral measure does not vanish at infinity, whence  $T_{\varphi, \mathcal{S}}|_{\mathcal{B} \otimes \mathcal{A}_{\Sigma_\varphi}}$  is not mixing. Then we can apply Proposition 11 and the result follows exactly as for Corollary 5.  $\square$

**Proposition 14 ([22])** *Assume that  $T$  is  $r$ -fold mixing and that  $\varphi$  is recurrent. If  $T_{\varphi, \mathcal{S}}$  is mildly mixing then it is also  $r$ -fold mixing.*  $\square$

Now, the corollary below directly follows from Proposition 14 and Proposition 12.

**Corollary 7** *Assume that  $T$  is  $r$ -fold mixing and  $\varphi$  is recurrent. Then  $T_{\varphi, \mathcal{S}}$  is  $r$ -fold mixing if and only if  $\sigma_{\mathcal{S}}(\Sigma_\varphi) = 0$ .*  $\square$

## References

- [1] J. Aaronson, *An Introduction to Infinite Ergodic Theory*, Math. Surveys and Monographs **50**, Amer. Math. Soc. 1997.
- [2] L.M. Abramov, V.A. Rokhlin, *The entropy of a skew product of measure preserving transformation*, Amer. Math. Soc. Transl. **48** (1965), 225-245.
- [3] H. Anzai, *Ergodic skew product transformation on the torus*, Osaka J. Math. **3** (1951), 83-99.
- [4] T. Austin, M. Lemańczyk, *Relatively finite measure-preserving extensions and lifting multipliers by Rokhlin cocycles*, Journal of Fixed Point Theory and Applications **6**, no 1 (2009), 115-131.
- [5] A. Danilenko, M. Lemańczyk, *A class of multipliers for  $\mathcal{W}^\perp$* , Israel J. Math. **148** (2005), 137-168.
- [6] H. Furstenberg, *Strict ergodicity and transformations of the torus*, Amer. J. Math. **83** (1961), 573-601.
- [7] H. Furstenberg, *Disjointness in ergodic theory, minimal sets and diophantine approximation*, Math. Syst. Th. **1** (1967), 1-49.
- [8] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton University Press, Princeton, New Jersey, 1981.
- [9] H. Furstenberg, B. Weiss, *The finite multipliers of infinite ergodic transformations*, Lect. Notes in Math. **668** (1978), 127-132.
- [10] S. Glasner, *Proximal Flows*, Lectures Notes in Math. **517**, 1976.
- [11] E. Glasner, *On the multipliers of  $\mathcal{W}^\perp$* , Ergodic Theory Dynam. Sys. **14** (1994), 129-140.
- [12] E. Glasner, *Ergodic Theory via Joinings*, Mathematical Surveys and Monographs **101**, AMS, Providence, RI, 2003.

- [13] E. Glasner, B. Weiss, *Processes disjoint from weak mixing*, Trans. Amer. Math. Soc. **316** (1989), 689-703.
- [14] F. Hahn, W. Parry, *Some characteristic properties of dynamical systems with quasi-discrete spectra*, Math. Systems Theory **2** (1968), 179-190.
- [15] T. Hamachi, M. Osikawa, *Ergodic groups of automorphisms and Krieger's theorems*, Seminar on Math. Sci. Keio Univ. **3** (1991).
- [16] B. Host, J.-F. Méla, F. Parreau, *Analyse Harmonique des mesures*, Astérisque **135-136**, SMF, 1986
- [17] B. Host, J.-F. Méla, F. Parreau, *Non-singular transformations and spectral analysis of measures*, Bull. Soc. Math. France **119** (1991), 33-90.
- [18] A. B. Katok in collaboration with E. A. Robinson Jr., *Cocycles, Cohomology and Combinatorial Constructions in Ergodic Theory*, Proceedings of Symposia in Pure Mathematics **69** (2001), 107-173.
- [19] A. Katok, J.-P. Thouvenot, *Spectral Properties and Combinatorial Constructions in Ergodic Theory*, Handbook of dynamical systems. Vol. 1B, 649-743, Elsevier B. V., Amsterdam, 2006.
- [20] M. Lemańczyk, *Spectral Theory of Dynamical Systems*, Encyclopedia of Complexity and System Science, Springer-Verlag (2009), 8554-8575.
- [21] M. Lemańczyk, E. Lesigne, *Ergodicity of Rokhlin cocycles*, J. d'Anal. Math. **85** (2001), 43-86.
- [22] M. Lemańczyk, F. Parreau, *Rokhlin extensions and lifting disjointness*, Ergodic Theory Dynam. Systems **23** (2003), 1525-1550.
- [23] M. Queffélec, *Substitution Dynamical Systems - Spectral Analysis*, Lecture Notes in Math. **1294**, 1987.
- [24] E.A. Robinson, *A general condition for lifting theorems*, Trans. Amer. Math. Soc. **330** (1992), 725-755.
- [25] D. Rudolph, *Classifying isometric extensions of a Bernoulli shift*, Israel J. Math. **34** (1978), 36-60.
- [26] D. Rudolph, *k-fold mixing lifts to weakly mixing isometric extensions*, Ergodic Theory Dynam. Systems **5** (1985), 445-447.
- [27] D. Rudolph,  *$\mathbb{Z}^n$  and  $\mathbb{R}^n$  cocycle extensions and complementary algebras*, Ergodic Theory Dynam. Systems **6** (1986), 583-599.
- [28] V.V. Ryzhikov, *Joinings, intertwining operators, factors, and mixing properties of dynamical systems*, Russian Acad. Sci. Izv. Math. **42** (1994), 91-114.
- [29] K. Schmidt, *Cocycles of Ergodic Transformation Groups*, Lecture Notes in Mathematics, Vol. 1, MacMillan Co. of India, 1977.
- [30] R. Zimmer, *Extensions of ergodic group actions*, Illinois J. Math. **20** (1976a), 373-409.

Mariusz Lemańczyk  
Faculty of Mathematics and Computer Science  
Nicolaus Copernicus University

12/18 Chopin street, 87–100 Toruń, Poland  
mlem@mat.uni.torun.pl

François Parreau  
Laboratoire d'Analyse, Géométrie et Applications, UMR 7539  
Université Paris 13 et CNRS  
99, av. J.-B. Clément, 93430 Villetaneuse, France  
parreau@math.univ-paris13.fr